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SPECTRUM, HARMONIC FUNCTIONS, AND HYPERBOLIC METRIC SPACES

BY

YURI KIFER*

Institute of Mathematics
The Hebrew University of Jerusalem, Jerusalem, Israel

ABSTRACT

The main result of the paper says, in particular, that if M is a complete simply connected Riemannian manifold with Ricci curvature bounded from below and without focal points, which is also a hyperbolic metric space in the sense of Gromov, then the top λ of the L^2-spectrum of the Laplace–Beltrami operator Δ is negative, the Martin boundary of M corresponding to Δ is homeomorphic to the sphere at infinity $S(\infty)$, and the harmonic measures on $S(\infty)$ have positive Hausdorff dimensions. These generalize the results of [AS], [An1], [Ki], [KL] and [BK]. Moreover, if $\dim M = 2$, then in the presence of the other conditions the hyperbolicity is also necessary for $\lambda < 0$. The machinery consists of a combination of geometrical and probabilistic means.

1. Introduction

Let M be a complete, simply connected n-dimensional, $n \geq 2$, C^3 Riemannian manifold without focal points which will be called here a generalized Cartan–Hadamard (CH) manifold, reserving the name CH-manifold for the case when all sectional curvatures of M are nonpositive. Then there is a natural geometric compactification of M by the sphere at infinity $S(\infty)$ which is the set of classes of asymptotic geodesics (see [Go2]). Let Δ denote the Laplace–Beltrami operator on M; then C^2-functions h on M satisfying $\Delta h = 0$ are called harmonic. It is known

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(see [AS], [An1], [Ki]) that if all sectional curvatures of M are sandwiched between two negative constants, then the space of minimal positive harmonic functions coincides with the Martin boundary of M (constructed for the operator Δ) which turns out to be homeomorphic to $S(\infty)$. Furthermore, the Dirichlet problem at infinity has a unique solution, i.e. there is a one-to-one correspondence between continuous functions f on $S(\infty)$ and harmonic functions h_f having a continuous extension to $S(\infty)$ given by the formula

$$h_f(x) = \int_{S(\infty)} f(\zeta)P(x, d\zeta) \tag{1.1}$$

where $P(x, \cdot)$, $x \in M$ are probability measures on $S(\infty)$, called the harmonic measures.

It is known (see [Eb2]) that if M has nonpositive sectional curvature, then M has no focal points and that the "no focal points" condition is equivalent to the strict growth in t of the norm $||J(t)||$ for any Jacobi field $J(t)$, $J(0) = 0$, $J'(0) \neq 0$ on each geodesic $\gamma(t)$, $t \geq 0$. Let S_xM be the sphere of unit vectors from the tangent space T_xM of M at x. By [Go2] the "no focal points" assumption implies that the map $\Phi_x : S_xM \to S(\infty)$, which maps a vector $\zeta \in S_xM$ to the end at ∞ of the geodesic with initial velocity ζ, is a homeomorphism. It turns out (see [Ka], [L]) that, in general, the measure $\nu_x = \Phi_x^{-1}P(x, \cdot)$ on S_xM is singular with respect to the Lebesgue measure on S_xM. Nevertheless, if all sectional curvatures of M are sandwiched between two negative constants, the Hausdorff dimension of ν_x is positive (see [KL]).

The probabilistic approach of [Ki] and [KL] did not need explicitly the curvature assumptions above, but required that M has bounded geometry, geodesics diverge exponentially fast, and $\Delta\rho \geq \text{const} > 0$ with $\rho(x) = \text{dist}(x, x_0)$. In [BK], which treats the case when M is the universal cover of a compact surface of nonpositive curvature, we showed that after some modification the method works under somewhat relaxed conditions allowing one to consider certain classes of manifolds with nonpositive rather than negative curvature. In this paper I make another step forward considering manifolds without focal points which are also hyperbolic metric spaces in the sense of Gromov. The latter means that there exists $\delta > 0$ such that for any geodesic triangle each point lying on one side is within distance δ from the union of two other sides (see, for instance, [Gro], [CDP]). I will call such manifolds hyperbolic. If M has nonpositive curvature, then M is hyperbolic if any only if it satisfies the Uniform Visibility Axiom from [Eb1].

Let $p(t,x,y)$ be the heat kernel on M (see [Ch]), i.e. the minimal positive fundamental solution of the parabolic equation $\partial p/\partial t = \Delta p$ where Δ is applied in the x-variable. The following is the main result of this paper.

THEOREM A: *Let M be a hyperbolic generalized* CH*-manifold with the Ricci curvature bounded from below. Then there exists $c > 0$ such that for any $x, y \in M$ and $t > 0$,*

$$p(t,x,y) \le c^{-1}t^{-n/2}\min(\exp(-ct), \exp(-cd(x,y))) \tag{1.2}$$

where d is the distance function on M. It follows that if λ is the top of the L^2-spectrum of Δ (see [Ch]*), then $\lambda \le -c$.*

Theorem A will be proved by a combination of geometrical considerations of Section 2 and the probabilistic machinery of Section 3. If the sectional curvature of M is sandwiched between two negative constants, then the positivity of $-\lambda$ is known by [MK]. In fact, the hyperbolicity is not necessary for Theorem A and I prove it under the more general K-condition (see Section 2) which implies uniform exponentially fast volume growth of all balls and is equivalent to the hyperbolicity when $\dim M = 2$ but in the multidimensional case allows, for instance, to have some imbedded flat planes. Moreover, the K-condition implies not just (1.2) but also stronger results of Section 3 about the radial behavior of the Brownian motion on M which are needed, in particular, in Section 4 for the proof of Theorem B. I give several sufficient conditions for the K-condition to be satisfied both in terms of the curvature and in terms of the volume growth. In particular, for the two-dimensional case these lead to a necessary and sufficient condition for the positivity of $-\lambda$.

ASSERTION: *Let M be a generalized CH-manifold with $\dim M = 2$ and the Ricci curvature bounded from below. Set $v(r) = \inf_{x\in M} m(B_x(r))$ where $B_x(r)$ is the ball of radius r centered at x. Then $\lambda < 0$ if and only if*

$$\lim_{r\to\infty} r^{-2}v(r) = \infty \tag{1.3}$$

which is equivalent in this case to the hyperbolicity of M.

In fact, by Proposition 2.9 and Corollary 2.17 for generalized CH-manifolds the equality (1.3) is equivalent to the uniform exponentially fast growth of volumes of all balls. It is plausible that also in the multidimensional case the latter is equivalent to the positivity of $-\lambda$. We remark that an exponentially fast growth of volumes of balls centered at one point does not suffice for the positivity of $-\lambda$, since in this case one still may have a sequence of Euclidean balls $B_{x_n}(r_n) \subset M$ with $r_n \to \infty$ which leads to $\lambda = 0$. In view of [An2] the following results are, mainly, the corollaries of Theorem A but I shall outline also their probabilistic proof which goes similarly to [Ki], [KL], and [BK].

THEOREM B: *Let M be as in Theorem A. Then the Dirichlet problem at infinity has a unique solution given by* (1.1). *Furthermore, the harmonic measures $P(x, \cdot)$ are positive on open subsets of $S(\infty)$, they have no atoms, and their Hausdorff dimensions are positive.*

THEOREM C: *Let M be as in Theorems A and B. Then the space of minimal positive harmonic functions on M coincides with the Martin boundary ∂M of M for the operator Δ and it is homeomorphic to $S(\infty)$. If, in addition, M satisfies the Uniform Visibility Axiom (see Section 2), in particular, if M has nonpositive sectional curvatures, then there exists a natural Hölder structure on ∂M and the above homeomorphism between ∂M and $S(\infty)$ is Hölder continuous together with its inverse.*

Recall that Martin's scheme requires the existence of, so-called, Green's function $G(x, y)$ for the operator Δ which is positive, harmonic in x for $x \neq y$, has certain singularity when $x \to y$, and, in our case, tends to zero when $\text{dist}(x, y) \to \infty$. Next, one studies limit points of the ratios

$$K(x, y_n) = \frac{G(x, y_n)}{G(x_0, y_n)}$$

for a sequence $\{y_n\}$ having no limit points in compact subdomains. This enables one to produce certain compactification whose boundary is called the Martin boundary. This boundary often coincides with, but in general includes, the space of minimal positive harmonic functions. Finally, every positive harmonic function can be represented as an integral over this space. In our circumstances, $G(x, y) = \int_0^\infty p(t, x, y)dt$ and it exists by (1.2).

In [An2] Ancona extended his potential theory method to show that if M is a hyperbolic manifold with bounded geometry and the Ricci curvature bounded from below, then the Martin boundary of M for the operator Δ is homeomorphic to its geometric boundary constructed by Gromov provided that the operator Δ is weakly coercive, i.e. the operator $\Delta + \varepsilon$ admits the Green function for some $\varepsilon > 0$. The key improvement given by Theorem A is that if M is a generalized CH-manifold with the Ricci curvature bounded from below, then hyperbolicity of M already implies weak coercivity of Δ, since (1.2) yields the Green function for any $\Delta + \varepsilon$ with $\varepsilon < c$. If there is a discrete group Γ of isometries of M such that the quotient $N = M/\Gamma$ is compact, then hyperbolicity of M implies that Γ is a hyperbolic group, and so it is nonamenable (cf. Assertion 5.3.A in [Gro]). Then by [Br] the top of the L^2-spectrum of Δ on M is negative, and so in this case Theorem A follows.

The key fact employed in the proof of Theorems A–C is the result from [Ca] (see also [Gro]) saying that the hyperbolicity of M is equivalent to a certain type of exponential divergence of geodesics similar to what was used in [BK]. It turns out that this implies also the positivity of $\Delta\rho$, $\rho(x) = d(x, x_0)$ on a "well spread" set (see Lemma 2.14) which enables us to keep control on the drift towards $S(\infty)$ of the Brownian motion on M and to go ahead with our probabilistic approach. The lower bound on the Ricci curvature provides upper bounds on $\Delta\rho$ which is the radial drift of the Brownian motion on M, and on the divergence rate of geodesics which is important in all known approaches to the problem. Note that I need neither bounded geometry nor upper curvature bounds assumptions for Theorems A–C. Actually, in view of [AC], a lower bound on the Ricci curvature together with the "no conjugate points" assumption yield the bounded geometry. If the "no focal points" condition is replaced by the "no conjugate points" assumption then $\Delta\rho$ may become negative at some points which complicates the situation. Still, some generalizations to the "no conjugate points" case are possible and they are discussed at the end of this paper.

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My probabilistic approach also suggests some generalizations which are discussed in Section 6. Some parts of this paper were written during my visit to the Max-Planck Institute in Bonn in April 1992 in course of the special activity on Stochastic Analysis and Geometry where I presented the main results of this paper. Some revisions were made in July 1992 during my stay at the University of Warwick and in September–October 1992 during my visit to the University of North Carolina at Chapel Hill.

2. Geometric preliminaries

Let M be a generalized CH-manifold. Then any two points can be joined by a unique geodesic. Following [Ca] a geodesic triangle in M will be called δ-thin if each point on one side is within distance δ from the union of two other sides. If there exists $\delta > 0$ such that any geodesic triangle is δ-thin, then M is called hyperbolic. Let $d(x,y)$ denotes the distance between $x, y \in M$. For each $x \in M$ set $B_x(r) = \{y \in M : d(x,y) < r\}$, $S_x(r) = \{y \in M : d(x,y) = r\}$, $B^c_x(r) = M \setminus B_x(r)$. For all $y, z \in B^c_x(r)$ put $d^r_x(y,z) = \inf\{\text{length } \gamma | \gamma : [0,a] \to M$ is a smooth curve, $\gamma \subset B^c_x(r)$, $\gamma(0) = y, \gamma(a) = z\}$. One says that geodesics emanating from x diverge C-exponentially fast, $C > 0$, if for any two geodesic segments γ_1 and γ_2 starting at x, any numbers r, ρ such that $0 < r < r + C \le \rho$, and any points $y_1, z_1 \in \gamma_1$ and $y_2, z_2 \in \gamma_2$ satisfying $y_1, y_2 \in S_x(r)$ and $z_1, z_2 \in S_x(\rho)$, it follows that

$$d^r_x(y_1, y_2) \le \frac{1}{2} d^\rho_x(z_1, z_2) + C. \tag{2.1}$$

Remark that if $d^r_x(y_1,y_2) \ge 3C$ then $d^\rho_x(z_1,z_2) \ge 2d^r_x(y_1,y_2) - 2C \ge \frac{4}{3} d^r_x(y_1,y_2)$, and so

$$d^\rho_x(z_1, z_2) \ge \left(\frac{3}{4}\right)^C \left(\frac{4}{3}\right)^{C^{-1}(\rho - r)} d^r_x(y_1, y_2), \tag{2.2}$$

i.e. (2.1) implies that the geodesics emanating from one point start diverging exponentially fast after they diverge $3C$ apart with respect to the distance d^r_x.

2.1. PROPOSITION (see [Ca]): (i) *If any geodesic triangle is δ-thin, then geodesics emanating from any point x diverge 11δ-exponentially fast.*

(ii) *If geodesics emanating from any point x diverge C-exponentially fast, then each geodesic triangle is $34C$-thin.*

Let M be a generalized CH-manifold. Then for any $x \in M$ the exponential map $\mathrm{Exp}_x : T_xM \to M$ is a diffeomorphism, where T_xM denotes the tangent space at x. Hence there is a global system of geodesic polar coordinates which assigns to each $z \in M$ the pair (r, ξ) where $r = d(x, z)$ and $\xi \in S_xM = \{\zeta \in T_xM | \|\zeta\| = 1\}$. For each $\xi \in S_xM$, let $\xi^\perp$ be the orthogonal complement of the one-dimensional subspace of T_xM generated by ξ, and let $\tau_t : T_xM \to T_{\mathrm{Exp}_x t\xi}M$ denote the parallel translation along the geodesic γ_ξ with $\gamma_\xi(0) = x$ and $\dot\gamma_\xi(0) = \xi$. Following [Ch] define the path of linear transformations $\mathcal{A}_x(t, \xi) : \xi^\perp \to \xi^\perp$ by $\mathcal{A}_x(t, \xi)\eta = \tau_t^{-1}J(t)$, where $J(t)$ is the Jacobi field along γ_ξ determined by the initial conditions $J(0) = 0$, $J'(0) = \eta$. Then (see [Ch]) one can write the Riemannian metric form in geodesic polar coordinates in the following way:

$$ds^2 = dr^2 + |\mathcal{A}_x(r, \xi)d\xi|^2.$$

Thus, in these coordinates the Laplace–Baltrami operator has the form

$$\Delta = \frac{\partial^2}{\partial r^2} + Q_x(r, \xi)\frac{\partial}{\partial r} + \Delta_\xi \tag{2.3}$$

where Δ_ξ does not contain derivatives in r,

$$Q_x(r, \xi) = \frac{\partial}{\partial r}(\log A_x(r, \xi)) = \Delta\rho_x(r), \tag{2.4}$$

ρ_x is the distance function from x, and $A_x(r, \xi) = \det \mathcal{A}_x(r, \xi)$ is the $(n-1)$-dimensional volume element on $S_x(r)$ at the point (r, ξ).

2.2. Lemma: *Let M be a generalized n-dimensional CH-manifold. Then for all $x \in M$, $\xi \in S_xM$, and $r > 0$,*

$$Q_x(r, \xi) > 0. \tag{2.5}$$

Suppose, in addition, that the Ricci curvature of M is bounded from below by $-k^2(n-1)$. Then

$$Q_x(r, \xi) \le C(r) \overset{\mathrm{def}}{=} k(n-1)\coth(kr) = (n-1)k(e^{kr} + e^{-kr})(e^{kr} - e^{-kr})^{-1} \tag{2.6}$$

and if $J(t)$ is any perpendicular Jacobi field along the geodesic γ_ξ, $\gamma_\xi(0) = x$, $\dot\gamma_\xi(0) = \xi$ such that $J(0) = 0$, $\|J'(0)\| = 1$, then for all $t > 0$,

$$\frac{d}{dt}\log\|J(t)\| \le C(t), \tag{2.7}$$

and so for all $t \geq s > 0$,

$$\|J(t)\| \leq \|J(s)\| \exp\left(\int_s^t C(u)du\right) \leq \|J(s)\| e^{(t-s)C(s)}.$$

It follows also that for any $\rho \geq r > 0$, and every curve $\sigma \subset S_x(r)$,

$$\text{length } \pi_x^\rho \sigma \leq \exp\left(\int_r^\rho C(u)du\right) \text{length } \sigma \leq e^{(\rho-r)C(r)} \text{length } \sigma \tag{2.8}$$

where $\pi_x^\rho y = \mathrm{Exp}_x(\rho r^{-1}\mathrm{Exp}_x^{-1} y)$ is the projection of $y \in S_x(r)$ to $S_x(\rho)$ along geodesics emanating from x. In particular, for any $\xi, \eta \in S_xM$ and $\rho \geq r > 0$,

$$d_x^\rho(\mathrm{Exp}_x \rho\xi, \mathrm{Exp}_x \rho\eta) \leq \exp\left(\int_r^\rho C(u)du\right) d_x^r(\mathrm{Exp}_x r\xi, \mathrm{Exp}_x r\eta). \tag{2.9}$$

Proof: By [Ch], p. 69,

$$Q_x(t,\xi) = \sum_{n-1 \geq i \geq 1} \frac{d}{dt} \log \|J_i(t)\| \tag{2.10}$$

where $J_i(t)$, $i = 1, \ldots, n-1$ are Jacobi fields along γ_ξ determined by the initial conditions $J_i(0) = 0$, $J_i'(0) = e_i$ with $\{e_1, \ldots, e_{n-1}\}$ being an orthonormal basis of $\xi^\perp$. By [Eb2] the "no focal points" property implies the positivity of each derivative in the right hand side of (2.10), and so (2.5) follows. If the Ricci curvature is bounded from below by $-k^2(n-1)$, then by Bishop's comparison theorem (see [Ch]), (2.6) holds true. Since each term of the sum in the right hand side of (2.10) is positive, it follows that

$$\frac{d}{dt} \log \|J_i(t)\| \leq k(n-1) \coth(kt), \quad i = 1, \ldots, n-1.$$

This yields (2.7), since an orthonormal basis $\{e_1, \ldots, e_{n-1}\}$ of $\xi^\perp$ which provides the initial conditions for $J_i(t)$, $i = 1, \ldots, n-1$ can be chosen arbitrarily. Finally, (2.8) and (2.9) follow from (2.7). Note that one can translate both the statement and the proof of this lemma into the language of the operator $U_x(r,\xi) = \mathcal{A}_x'(r,\xi)\mathcal{A}_x^{-1}(r,\xi)$ of the second fundamental form of $S_x(r)$ (see [Ch] and [Gra]). Then $Q_x(r,\xi) = \mathrm{tr} U_x(r,\xi)$ (tr denotes trace) and we derive (2.5)–(2.7) from the convexity of $S_x(r)$ (see [Eb3]) and Bishop's comparison theorem. ∎

Let M be a generalized CH-manifold. For any $x, y \in M$ there is a unique geodesic γ_{xy} such that $\gamma_{xy}(0) = x$ and $\gamma_{xy}(t) = y$ where $t = d(x,y)$. Let

$\angle_p$ denote the angle between vectors in the tangent space T_pM given by the Riemannian metric. For $p \in M$, $v \in T_pM$, $x, y \in M \setminus \{p\}$, $\Gamma \subset M \setminus \{p\}$ set

$$\angle_p(x,y) = \angle_p(\dot\gamma_{px}(0), \dot\gamma_{py}(0)), \angle(v,x) = \angle(v, \dot\gamma_{px}(0)),$$

and $\angle_p(\Gamma) = \sup\{\angle_p(x,y) | x, y \in \Gamma\}$.

2.3. LEMMA: *Let M be a generalized CH-manifold. Then for any $x \in M$ there exist $c_x > 0$ and an increasing positive function $q_x(t)$ on $[1,\infty)$ with $\lim_{t\to\infty} q_x(t) = \infty$ such that for any perpendicular Jacobi field $J(t)$, $J(0) = 0$, $||J'(0)|| = 1$ on each geodesic γ emanating from x,*

$$||J(t)|| \geq q_x(t)||J(1)|| \quad \text{and} \quad ||J(1)|| \geq c_x. \tag{2.11}$$

The constant c_x can be chosen so small that for any smooth curve $\sigma \subset S_x(1)$,

$$\text{length } \sigma \geq c_x \angle_x(\sigma). \tag{2.12}$$

It follows that for any smooth curve $\sigma \subset S_x(r)$, $r \geq 1$,

$$\text{length } \sigma \geq q_x(r)\text{length } \pi_x^1\sigma \geq c_x q_x(r) \angle_x(\sigma) \tag{2.13}$$

where π_x^s is the same as in Lemma 2.2. If the curvature of M is nonpositive, then one can take $q_x(t) = t$ and $c_x = 1$.

Proof: The existence of c_x satisfying (2.11) and (2.12) follows from compactness of closed balls $\overline{B_x(1)}$ and the continuous dependence of Jacobi fields on initial conditions. Set

$$q_x(t) = \inf_J (||J(t)|| ||J(1)||^{-1}), \quad t \geq 1,$$

where the infimum is taken over all perpendicular Jacobi fields $J(t)$, $J(0) = 0$, $||J'(0)|| = 1$ along geodesics emanating from x. Then $q_x(t)$ is an increasing in t function since in a generalized CH-manifold the norms $||J(t)||$ of all Jacobi fields, as above, grow in t. By [Go1], $\lim_{t\to\infty} q_x(t) = \infty$. We remark that this follows also from [Eb1] and [Eb2] under the additional assumption that sectional curvature of M is bounded from below but, in fact, the proof there requires only the uniform bound (2.7). The inequality (2.13) is a direct consequence of (2.11) and (2.12). If M has nonpositive curvature, then by the Rauch comparison theorem (see [GKM]) perpendicular Jacobi fields on each geodesic grow, at least, as fast as in the zero curvature case, and so one can take $q_x(t) = t$ and $c_x = 1$ for all $x \in M$. ∎

2.4. LEMMA: *Let M be a generalized CH-manifold. Then for all $x \in M$ and any smooth curve $\sigma : [0,a] \to M, \sigma \subset B_x^c(r)$,*

$$\text{length } \pi_x^r \sigma \leq \text{length } \sigma. \tag{2.14}$$

The equality in (2.14) holds true if and only if $\sigma \subset S_x(r)$.

Proof: Let $\delta > 0$ be a small number and $b_0 \in [0, a-\delta]$, so that $\tilde{\sigma} : [b_0, b_0+\delta] \to M$ is a small piece of the curve σ. Set $r_0 = \inf\{d(x,y) | y \in \tilde{\sigma}\}$ and suppose that $y_0 \in \tilde{\sigma}$ is the point where $r_0 = d(y_0, x)$. I claim that

$$\text{length } \pi_x^{r_0} \tilde{\sigma} \leq \text{length } \tilde{\sigma} + o(\delta). \tag{2.15}$$

Indeed, up to an error of order $o(\delta)$ one can pass to the metric

$$ds_0^2 = \sum_{i,j} g_{ij}(x_0) dx^i dx^j$$

freezing the coefficients of the original metric form $\sum_{i,j} g_{ij}(x) dx^i dx^j$ at x_0 and arriving at the linear problem of projecting $\tilde{\sigma}$ in R^d to a hyperplane Γ along parallel lines orthogonal to Γ with respect to the inner product generated by $(g_{ij}(x_0))$. Measuring the length with respect to the metric ds_0 we arrive easily at (2.15). Next, since $r_0 \geq r$ and the "no focal points" property implies the growth of all perpendicular Jacobi fields along geodesic emanating from x, it follows that

$$\text{length } \pi_x^r \tilde{\sigma} = \text{length } \pi_x^r \pi_x^{r_0} \tilde{\sigma} \leq \text{length } \pi_x^{r_0} \tilde{\sigma}$$

which yields (2.14). This inequality is strict if $r_0 > r$. ∎

Following [Eb1] one says that M satisfies the Visibility Axiom if for any $\varepsilon > 0$ and $p \in M$ there exists a number $r(\varepsilon, p)$ such that, if $\gamma \subset B_p^c(\rho)$, $\rho \geq r(\varepsilon, p)$ is a geodesic segment, then $\angle_p(\gamma) \leq \varepsilon$. If $r(\varepsilon, p) = r(\varepsilon)$ is independent of p, then M is said to satisfy the Uniform Visibility Axiom.

2.5. PROPOSITION: *Let M be a generalized CH-manifold.*

(i) *If M satisfies the Uniform Visibility Axiom with $r(\varepsilon)$, then all geodesic triangles are $r(\delta\frac{\pi}{2})$-thin for any $\delta \in (0,1)$. Furthermore, let $\sigma \subset S_p(\rho)$ be a smooth curve. Then*

$$\text{length } \sigma \geq 2(\rho - r(\delta \angle_p \sigma)) \quad \textit{for any } \delta \in (0,1).$$

(ii) *If all geodesic triangles in M are δ-thin, then M satisfies the Visibility Axiom with some $r(\varepsilon, p)$. If, in addition, M has nonpositive sectional curvature, then it satisfies the Uniform Visibility Axiom with $r(\varepsilon) = 4\delta\varepsilon^{-1}$.*

Proof:

(i) Let $\{A, B, C\}$ be a geodesic triangle with vertices A, B, C and let x be an arbitrary point on the geodesic segment γ_{BC}. If for some $\varepsilon > 0$ both $d(x, \gamma_{AB}) \geq r(\varepsilon)$ and $d(x, \gamma_{AC}) \geq r(\varepsilon)$, then $\max(\angle_x(A, B), \angle_x(A, C)) \leq \varepsilon$. Since $\angle_x(A, B) + \angle_x(A, C) = \pi$ then $\varepsilon \geq \frac{\pi}{2}$, and so $\{A, B, C\}$ is $r(\delta\frac{\pi}{2})$-thin if $0 < \delta < 1$. Next, let $\sigma \subset S_p(\rho)$ be a smooth curve with endpoints y and z. Then the geodesic segment γ_{yz} satisfies $d(p, \gamma_{yz}) \geq \rho - \frac{1}{2}\text{length } \gamma_{yz}$, and so $r(\delta\angle_p\sigma) \geq \rho - \frac{1}{2}\text{length}\gamma_{yz} \geq \rho - \frac{1}{2}\text{length } \sigma$ if $0 < \delta < 1$, completing the proof of (i).

(ii) Let $d(p, \gamma_{AB}) = r$ for some points A, B, and p. Consider the geodesic triangle $\{p, A, B\}$ and take $x \in \gamma_{pA}$ such that $d(p, x) = r/2$. Then $d(x, \gamma_{AB}) \geq r/2$. Let $d(x, \gamma_{pB}) = \tilde{r}$ and $y \in \gamma_{pB}$ satisfies $d(x, y) = \tilde{r}$. If $\gamma_{xy} \cap B_{r/4}(p) \neq \emptyset$ then $\tilde{r} \geq r/4$. If $\gamma_{xy} \subset B^c_{r/4}(p)$ then, by Lemmas 2.3 and 2.4,

$$\tilde{r} = \text{length } \gamma_{xy} \geq c_p q_p(r/4)\angle_p(\gamma_{AB}).$$

Since $\{p, A, B\}$ is δ-thin, then for $r > 4\delta$, $\angle_p(\gamma_{AB}) \leq \delta c_p^{-1}(q_p(r/4))^{-1}$, and so one can take $r(\varepsilon, p) = 4q_p^{-1}(\delta c_p^{-1}\varepsilon^{-1})$ where q_p^{-1} is the inverse function for q_p. If M has nonpositive curvature then, by Lemma 2.3, $c_p = 1$ and $q_p(t) = t$, and so $r(\varepsilon, p) = r(\varepsilon) = 4\delta\varepsilon^{-1}$. ∎

Let $\mathcal{C}_p(\xi, \delta) = \text{Exp}_p\{\eta \in T_pM : \angle_p(\xi, \eta) < \delta\}$ denote the cone about $\xi \in S_pM$ of angle δ with vertex at p.

2.6. Lemma: *Let M be a generalized CH-manifold satisfying the Visibility Axiom with some $r(\varepsilon, p)$. Then for any $t \geq r(\varepsilon, p)$,*

$$\mathcal{C}_p(\xi, \varepsilon) \supset \mathcal{C}_{\gamma_\xi(t)}\left(\dot{\gamma}_\xi(t), \frac{\pi}{2}\right) \tag{2.16}$$

where $\gamma_\xi(t)$ is the geodesic satisfying $\gamma_\xi(0) = p$ and $\dot{\gamma}_\xi(0) = \xi \in S_pM$.

Proof: Since $\gamma_\xi(t)$ is orthogonal to $S_p(t)$ at $q = \gamma_\xi(t)$ (see Lemma 10.5 in [Mi]), then for any $\eta \in S_qM$ satisfying $\angle_q(\dot{\gamma}_\xi(t), \eta) < \frac{\pi}{2}$ there exists $s > 0$ such that, if γ_η is the geodesic with $\gamma_\eta(0) = q$, $\dot{\gamma}_\eta(0) = \eta$, then $\gamma_\eta(u) \notin \overline{B_p(t)}$ for all $u \in (0, s)$. Let $s(\eta)$ denote the supremum of such $s > 0$. If $s(\eta) < \infty$ then $\gamma_\eta(s(\eta)) \in S_p(t)$

and, by Lemma 2.4, $\gamma_\eta(u)$, $0 \le u \le s(\eta)$ cannot be a geodesic. Thus $s(\eta) = \infty$, and so $\gamma_\eta(u) \in B_p^c(t)$ for all $u \ge 0$. By the Visibility Axiom $\angle_p(\gamma_\eta) \le \varepsilon$ provided $t \ge r(\varepsilon, p)$, and so $\gamma_\eta \subset \mathcal{C}_p(\xi, \varepsilon)$ yielding (2.16). ∎

Proposition 2.5 together with the following result from [Eb1] may help the reader to clarify what kind of manifolds this paper deals with.

2.7. PROPOSITION (see [Eb1]): *Let M be the universal cover of a compact manifold N with a metric g lifted from N.*

(i) *If (M, g) has nonpositive curvature, then (M, g) satisfies the Uniform Visibility Axiom if and only if M contains no totally geodesic isometric imbedding of the Euclidean plane.*
(ii) *Suppose that M is a generalized CH-manifold with respect both to g and to another metric $\tilde{g}$ lifted from N. Then $(M, \tilde{g})$ satisfies the Uniform Visibility Axiom if and only if (M, g) does.*

We remark that if M is the universal cover of a compact manifold N with a metric lifted from N and it satisfies the Visibility Axiom, then it satisfies the Uniform Visibility Axiom. Proposition 2.7 gives the characterization of hyperbolic (generalized) CH-manifolds having a compact quotient. Apart from this case note that the hyperbolicity is preserved under quasiisometries (see [CDP]), and so those quasiisometries which do not destroy the "no focal points" property lead again to manifolds satisfying conditions of Theorems A, B and C. Some examples of manifolds without focal points can be found in [Gu].

For any $x \in M$, $r, \delta > 0$, and $y \in S_x(r)$ set

$$D_x^r(y, \delta) = \{z \in S_x(r) : d_x^r(y, z) \le \delta\}.$$

2.8. LEMMA: *Let M be a n-dimensional generalized CH-manifold with the Ricci curvature bounded from below by $-k^2(n-1)$. Then there exist two positive functions ψ, Ψ on $(0, \infty)$ such that for any $x \in M, \rho, r > 0$, and $y \in S_x(r)$,*

$$e^{-2\rho C(r)}\psi(\rho) \le m_x^r(D_x^r(y, \rho)) \le \Psi(\rho) \tag{2.17}$$

where m_x^r is the $(n-1)$-dimensional volume on $S_x(r)$ induced by the Riemannian metric and $C(r)$ is defined by (2.6).

Proof: Set $U = D_x^r(y, \rho)$; then by the triangle inequality

$$B_y(2\rho) \supset \bigcup_{r+\rho \ge t \ge r} \pi_x^t U. \tag{2.18}$$

It follows from (2.4) and (2.5) that for any $t \geq r$,

$$m_x^t(\pi_x^t(U)) \geq m_x^r(U), \tag{2.19}$$

which together with (2.18) gives

$$m(B_y(2\rho)) \geq \rho m_x^r(U).$$

By the volume comparison theorems (see [BC], Section 11.10 or [Gra], Section 3.5) $m(B_y(2\rho))$ does not exceed the volume of a ball of radius 2ρ in the space of constant curvature $-k^2$ yielding the upper bound in (2.17).

Next, I prove the left hand side of (2.17). By (2.14) and the triangle inequality

$$V = \bigcup_{r+2\rho \geq t \geq r} \pi_x^t U \supset B_v(\rho)$$

where $v = \mathrm{Exp}_x(r+\rho)\zeta$ and $\zeta = r^{-1}\mathrm{Exp}_x^{-1} y$. In the same way as above

$$2\rho m_x^{r+2\rho}(\pi_x^{r+2\rho} U) \geq m(V) \geq m(B_v(\rho)).$$

By (2.4) and (2.6),

$$m_x^{r+2\rho}(\pi_x^{r+2\rho} U) \leq e^{2\rho C(r)} m_x^r(U), \tag{2.20}$$

and so

$$m_x^r(U) \geq 1/2\rho^{-1} e^{-2\rho C(r)} m(B_v(\rho)). \tag{2.21}$$

From Proposition 14 of [Cr] (which is stated for compact manifolds but works for noncompact manifolds as well) it follows that

$$m(B_v(\rho)) \geq c_n \rho^n,$$

where c_n depends only on $n = \dim M$ which, together with (2.21), completes the proof of the lower bound in (2.17). ∎

I will say that M satisfies the K-condition, $K > 0$ if for any $x, y \in M$ there exist $r, s > 0$, $s \leq K$, and a Borel set $U \subset S_x(r) \cap B_y(K)$ such that

$$m_x^r(U) \geq K^{-1} \quad \text{and} \quad m_x^{r+s}(\pi_x^{r+s} U) \geq (1 + K^{-1}) m_x^r(U). \tag{2.22}$$

The following result says that the K-condition implies the uniform exponentially fast growth of volumes of all balls.

2.9. PROPOSITION: *Let M be a generalized CH-manifold with the Ricci curvature bounded from below satisfying the K-condition. Then there exists $\delta > 0$ such that, for any $x \in M$ and $r \geq \delta^{-1}$,*

$$m(B_x(r)) \geq e^{\delta r}. \tag{2.23}$$

Proof: Suppose that the K-condition holds true with some $K \geq 1$. Pick up some $x \in M$ and $\rho \geq K$. On the sphere $S_x(\rho)$ choose a maximal collection of points $y_1, y_2, \ldots, y_\ell$ so that $d_x^r(y_i, y_j) > 2K$ for any $i \neq j$. The word maximal means here that one cannot add points to this collection preserving the above condition. Then $V_i = D_x^\rho(y_i, K)$ are disjoint for different i's and

$$\bigcup_{i=1}^{\ell} D_x^\rho(y_i, 2K) \supset S_x(\rho). \tag{2.24}$$

Set $\alpha_i = \rho^{-1}\mathrm{Exp}_x^{-1} y_i$ and $z_i = \mathrm{Exp}_x(\rho + K)\zeta_i$. Then by (2.14),

$$B_{z_i}(K) \subset W_i = \bigcup_{\rho \leq t \leq \rho+2K} \pi_x^t V_i.$$

The K-condition provides r, s with $0 < s \leq K$, $\rho < r < \rho + 2K$ and $U \subset S_x(r) \cap B_{z_i}(K)$ satisfying (2.22). Set $\tilde{U} = \pi_x^\rho U$ and $\tilde{\tilde{U}} = \pi_x^{\rho+3K} U$. Then by (2.4)–(2.6) and (2.22),

$$m_x^\rho(\tilde{U}) \geq e^{-2KC(K)} m_x^r(U) \geq K^{-1} e^{-2KC(K)} \tag{2.25}$$

and

$$m_x^{\rho+3K}(\tilde{\tilde{U}}) \geq (1 + K^{-1} m_x^\rho(\tilde{U})). \tag{2.26}$$

Set $\tilde{V}_i = \pi_x^{\rho+3K} V_i$. Then by (2.4), (2.5), (2.17), (2.25), and (2.26),

$$\begin{aligned} m_x^{\rho+3K}(\tilde{V}_i) &= m_x^{\rho+3K}(\tilde{\tilde{U}}) + m_x^{\rho+3K}(\tilde{V}_i \setminus \tilde{\tilde{U}}) \\ &\geq (1 + K^{-1}) m_x^\rho(\tilde{U}) + m_x^\rho(V_i \setminus \tilde{U}) \\ &= (1 + K^{-1} m_x^\rho(\tilde{U}) / m_x^\rho(V_i)) m_x^\rho(V_i) \\ &\geq (1 + \beta) m_x^\rho(V_i) \end{aligned} \tag{2.27}$$

where $\beta = e^{-2KC(K)}K^{-1}(\Psi(K))^{-1}$. Similarly, by (2.17) and (2.24),

$$\begin{aligned} m_x^{\rho+3K}(S_x(\rho+3K)) &\geq m_x^{\rho+3K}(\bigcup_i \tilde{V}_i) + m_x^{\rho+3K}(S_x(\rho+3K)\setminus \bigcup_i \tilde{V}_i) \\ &\geq (1+\beta)m_x^\rho(\bigcup_i V_i) + m_x^\rho(S_x(\rho)\setminus\bigcup_i V_i) \\ &\geq m_x^\rho(S_x(\rho))(1+\beta m_x^\rho(\bigcup_i V_i)/m_x^\rho(S_x(\rho))) \\ &\geq m_x^\rho(S_x(\rho))(1+\beta L) \end{aligned} \tag{2.28}$$

where $L = e^{-2kC(1))}\psi(K)(\Psi(2K))^{-1}$.Then (2.23) holds true with

$$\delta = (9K)^{-1}\min(1, \log(1+\beta L)). \qquad \blacksquare$$

2.10. LEMMA:

(i) *Let M be a hyperbolic generalized CH-manifold with the Ricci curvature bounded from below. Then M satisfies the K-condition with some $K > 0$.*

(ii) *Let M be a 2-dimensional generalized CH-manifold with the Ricci curvature bounded from below and satisfying the K-condition. Then M is hyperbolic.*

Proof: By hyperbolicity there exists $C > 0$ such that geodesics emanating from any $x \in M$ diverge C-exponentially fast. Consider $D_x^r(y, 3C)$ with $r \geq 2C$. Let $\gamma(t) = \gamma_{xy}(t)$ for $0 \leq t \leq r$ and $\gamma(-r) = z \in S_x(r)$ be a geodesic segment. Then $d_x^r(y,z) \geq d(y,z) = 2r$, and so $z \notin D_x^r(y,3C)$. Hence $D_x^r(y,3C)$ is a proper subset of $S_x(r)$, and so the boundary $\partial D_x^r(y,3C)$ is not empty. Set ℓ_0 = integral part of $(1+2L\Psi(3C)(\psi(3C))^{-1})$ where $L = e^{6C(2C)C}$ and C comes from (2.1) while $C(\rho)$ comes from (2.6). If $r_0 = ((\log\frac{4}{3})^{-1}\log\ell_0 + C)C$ then by (2.2) with $\rho = r + r_0$ for any point $v \in \partial D_x^r(y,3C)$, $d_x^\rho(\pi_x^\rho y, \pi_x^\rho v) \geq 3C\ell_0$. Thus one can pick up points $w_0 = \pi_x^\rho y$, $w_1, w_2 \ldots, w_{\ell_0-1}$ in $\pi_x^\rho D_x^r(y,3C)$ such that $d_x^\rho(w_{i-1}, w_i) = 3C$ for all $i = 1, 2, \ldots, \ell_0 - 1$, and so that

$$D_x^\rho(w_i, C) \subset \pi_x^\rho D_x^r(y,3C), \quad i = 0, \ldots, \ell_0 - 1$$

and $D_x^\rho(w_i,C) \cap D_x^\rho(w_j,C) = \emptyset$ for all $0 \leq i < j \leq \ell_0 - 1$. Then by (2.17),

$$\begin{aligned} m_x^\rho(\pi_x^\rho D_x^r(y,3C)) &\geq \ell_0 L^{-1}\psi(3C) \\ &\geq \ell_0 L^{-1}\psi(3C)(\Psi(3C))^{-1} m_x^r(D_x^r(y,3C)) \\ &\geq 2m_x^r(D_x^r(y,3C)). \end{aligned} \tag{2.29}$$

Now take any $x, \tilde{y} \in M$ and set $r = \max(2C, d(x, \tilde{y}))$, $y = \pi_x^r \tilde{y}$ if $\tilde{y} \neq x$ and y is any point on $S_x(2C)$ if $\tilde{y} = x$. Repeat the above construction with such x, y, and r. Then

$$\bigcup_{r+r_0 \geq t \geq r} \pi_x^t D_x^r(y, 3C) \subset B_{\tilde{y}}(5C + r_0). \tag{2.30}$$

By (2.29) and (2.30), M satisfies the K-condition with

$$K = \max(5C + r_0 + 1, L(\psi(3C))^{-1}).$$

(ii) Let now M be a 2-dimensional generalized CH-manifold satisfying the K-condition, $K > 1$. Pick up an arbitrary $x \in M$ and $z, v \in S_x(r)$ with some $r \geq K$ so that $d_x^r(z, v) = 2\rho$ and $2K \geq \rho \geq K$. Let $w \in S_x(r)$ satisfy $d_x^r(w, z) = d_x^r(w, v) = \rho$. Set $\zeta = r^{-1}\mathrm{Exp}_x^{-1} w$ and $y = \mathrm{Exp}_x(r + K)\zeta$. It follows from (2.14) that

$$B_y(K) \subset \bigcup_{r+2K \geq s \geq r} \pi_x^s D_x^r(w, \rho).$$

By the K-condition there exists $s \in (r, r + 2K)$ and a Borel set $U \subset \pi_x^s D_x^r(w, \rho)$ such that $m_x^s(U) \geq K^{-1}$ and $m_x^{s+u}(\pi_x^{s+u} U) \geq (1 + K^{-1}) m_x^s(U)$ for some $u \in (0, K]$. By (2.8), $m_x^r(\pi_x^r U) \geq K^{-1} e^{-2KC(1)}$ and, clearly, $m_x^r(D_x^r(w, \rho)) = 2\rho \leq 4K$. Thus if $\Xi = r^{-1}\mathrm{Exp}^{-1} D_x^r(w, \rho)$ and $\Xi_u = s^{-1}\mathrm{Exp}_x^{-1} U$ then

(2.31)

$$\begin{aligned} d_x^{r+2K}(\pi_x^{r+2K} z, \pi_x^{r+2K} v) &= \int_{\Xi} A_x(r + 2K, \xi) d\xi \\ &\geq \int_{\Xi \setminus \Xi_u} A_x(r, \xi) d\xi + (1 + K^{-1}) \int_{\Xi_u} A_x(r, \xi) d\xi \\ &= m_x^r(D_x^r(w, \rho))(1 + K^{-1} \frac{m_x^r(\pi_x^r U)}{m_x^r(D_x^r(w, \rho))}) \\ &\geq d_x^r(z, v)(1 + \frac{1}{4} K^{-3} e^{-2KC(1)}). \end{aligned}$$

This, clearly, implies an exponentially fast divergence of geodesics and, by Proposition 2.1, yields the hyperbolicity of M. ∎

Note that the K-condition is weaker then the hyperbolicity assumption. The K-condition represents a certain growth condition on surface areas of geodesic spheres in M. It may hold true even when M contains flat planes. For instance, the K-condition always holds true if M is the universal cover of a compact rank 1 manifold of nonpositive curvature (see Lemma 2.1 in [BK]).

The following result provides a sufficient condition in terms of the curvature for the K-condition to hold true.

2.11. LEMMA: *Let M be a generalized CH-manifold with the Ricci curvature bounded from below. Suppose that there exists $L > 0$ such that for any $x, y \in M$ one can pick up $r, s > 0$, $s \leq L$, and a Borel set $U \subset S_x(r) \cap B_y(L)$ satisfying*

$$m_x^r(U) \geq L^{-1} \quad \text{and} \quad \int_r^{r+s} \rho_{\gamma_\xi(t)}(\dot{\gamma}_\xi(t))dt \leq -L^{-1} \tag{2.32}$$

provided $\xi = r^{-1}\mathrm{Exp}_x^{-1} z$ and $z \in U$ where $\rho_v(\zeta)$ denotes the Ricci curvature in the direction $\zeta \in S_v M$. Then the K-condition holds true with some $K = K(L) > 0$ specified in the proof.

Proof: Recall that $Q_x(t,\xi) = \mathrm{tr} U_x(t,\xi)$ where $U_x(t,\xi)$ is the operator of the second fundamental form of $S_x(t)$ at $v = (t,\xi)$ which satisfies the matrix Riccati equation (see [Ch], p.72),

$$\frac{\partial}{\partial t} U_x(t,\xi) + (U_x(t,\xi))^2 + \mathcal{R}_x(t,\xi) = 0 \tag{2.33}$$

where $\mathcal{R}_x(t,\xi) = \tau_t^{-1}\mathbf{R}_x(t,\xi)\tau_t$, $\mathbf{R}_x(t,\xi)$ is the curvature operator, and τ_ρ is the parallel translation along γ_ξ. The operator $U_x(t,\xi)$ is self-adjoint and the "no focal points" assumption implies that $U_x(t,\xi)$ is positive definite for all $t > 0$ and $\xi \in S_x M$ since all spheres are convex. Then $(\mathrm{tr} U_x(t,\xi))^2 \geq \mathrm{tr}(U_x(t,\xi))^2$ and, since $\mathrm{tr}\mathcal{R}_x(t,\xi) = \rho_{\gamma_\xi(t)}(\dot{\gamma}_\xi(t))$, one derives from (2.5), (2.32) and (2.33) that

$$\begin{aligned} Q_x(r+L,\xi) &= Q_x(r,\xi) - \int_r^{r+L} \mathrm{tr}(U_x(t,\xi))^2 dt - \int_r^{r+L} \rho_{\gamma_\xi(t)}(\dot{\gamma}_\xi(t)) \\ &\geq L^{-1} - \int_r^{r+L} (Q_x(t,\xi))^2 dt \end{aligned} \tag{2.34}$$

provided $\mathrm{Exp}_x(r\xi) \in U$. It follows that for any such ξ there exists $t = t(\xi) \in [r, r+L]$ such that

$$Q_x(t,\xi) \geq \frac{1}{2} L^{-1}(\sqrt{5} - 1) \stackrel{\mathrm{def}}{=} a(L). \tag{2.35}$$

On the other hand, by the Cauchy–Schwarz inequality

$$\mathrm{tr}(U_x(t,\xi))^2 \geq (\mathrm{tr} U_x(t,\xi))^2 (n-1)^{-1}.$$

Thus if $\rho_v(\zeta) \geq -k^2(n-1)$ for all $v \in M$ and $\zeta \in S_vM$, then taking trace in both parts of (2.33) one concludes by (2.6) that for all $t \geq r-1$,

$$\begin{aligned}\frac{\partial}{\partial t}Q_x(t,\xi) &\leq k^2(n-1) + (Q_x(t,\xi))^2(n-1)^{-1} \\ &\leq k^2(n-1) + (C(r-1))^2(n-1)^{-1} \stackrel{\text{def}}{=} b(r-1).\end{aligned} \tag{2.36}$$

Without loss of generality I can assume that $r \geq 2$, since if $d(x,y) < L+2$ then pick up $\tilde{y} \in S_x(L+2)$ and take $r, s, U \subseteq S_x(r)$ as in the statement of Lemma 2.10 for the pair $x, \tilde{y}$ in place of x, y. Then $U \subset B_y(3L+3)$. Now (2.4), (2.35) and (2.36) with $r \geq 2$ yield

$$\begin{aligned}\log(A_x(r+L,\xi)(A_x(r-1,\xi))^{-1}) &= \int_{r-1}^{r+L} Q_x(t,\xi)dt \\ &\geq \frac{a(L)}{2}\min(a(L)/b(1),1) \stackrel{\text{def}}{=} c(L).\end{aligned} \tag{2.37}$$

Thus

$$m_x^{r+L}(\pi_x^{r+L}(U)) \geq e^{c(L)} m_x^{r-1}(\pi_x^{r-1}U) \tag{2.38}$$

and the K-condition follows with $K = \max(3L+3, (e^{c(L)}-1)^{-1}, Le^{C(1)})$. ∎

2.12. Remark: After preparing the first draft of this paper I was informed by D. Elworthy that S. Kotani suggested recently another condition which enables him to prove the positivity of the bottom of the spectrum of $-\Delta$. I thank S. Kotani who sent me the precise statement of his result. He assumes that M is a generalized CH-manifold with $\rho_x(\xi) \leq 0$ for all $x \in M$ and $\xi \in T_xM$ and that there exist $x \in M$ and $T > 0$ such that

$$\alpha(x,T) = \inf_{\xi \in T_xM, t \geq 0} \int_t^{t+T} (-\rho_{\gamma_\xi(s)}(\dot{\gamma}_\xi(s)))ds > 0. \tag{2.39}$$

Then he shows that $Q_x(r,\xi) \geq k_0(T) = (T + (\alpha(x,T))^{-1})^{-1}$ and, by standard comparison theorems (see [IW]), it follows that the bottom of the spectrum of $-\Delta$ is not less than $\frac{1}{4}(k_0(T))^2$. It is clear that (2.39) is substantially stronger than the assumption of Lemma 2.11, since the former does not allow an infinite geodesic staying entirely in a flat region though the latter includes many such cases. Also, the proof of Theorem A in Section 3 under the K-condition is more complicated because in this case one does not have a positive uniform lower bound for $Q_x(r,\xi)$. ∎

2.13. COROLLARY: *Let M be a CH-manifold with Ricci curvature bounded from below. Suppose that there exists $\delta > 0$ such that for any $z \in M$ the ball $B_z(\delta^{-1})$ contains a Borel set G_z such that $m(G_z) \geq \delta$ and, for any $v \in G_z$ and $\zeta \in S_vM$, $\rho_v(\zeta) \leq -\delta$. Then the assumption of Lemma 2.11 holds true with some $L = L(\delta) > 0$ and so the K-condition is satisfied.*

Proof: Pick up $x, y \in M$ arbitrarily. Choose $w \in S_x(\delta^{-1})$ so that $y \in \gamma_{xw}$ if $y \in B_x(\delta^{-1})$ and $w = y$ if $y \notin B_x(\delta^{-1})$. If $r = d(x,w)$ set $\zeta = r^{-1}\text{Exp}_x^{-1}w$ and $z = \text{Exp}_x(r + \delta^{-1})\zeta$. Since M has nonpositive curvature, then by comparison with the Euclidean space one concludes that

$$W = \bigcup_{r+2\delta^{-1} \geq s \geq r} \pi_x^s D_x^r(w, \delta^{-1})$$

is contained in the cone $\mathcal{C}_x(\zeta, 1)$. By (2.14) and the triangle inequality it follows that

$$B_y(4\delta^{-1}) \supset W \supset B_z(\delta^{-1}) \supset G_z. \tag{2.40}$$

Set $V = D_x^r(w, \delta^{-1})$, $\Xi = r^{-1}\text{Exp}_x^{-1}V$, and $\rho_v = \sup_{\zeta \in S_vM} \rho_v(\zeta)$. By (2.4) and (2.6) for any $t \geq r \geq \delta^{-1}$,

$$A_x(t,\xi)/A_x(r,\xi) \leq e^{(t-r)C(\delta^{-1})}. \tag{2.41}$$

Since $\rho_v \leq 0$ for all v, then by (2.40) and (2.41),

$$\begin{aligned}
-\delta^2 \geq \int_W \rho_v dm(v) &\geq e^{-2\delta^{-1}C(\delta^{-1})} \int_\Xi A_x(r,\xi)d\xi \int_r^{r+2\delta^{-1}} \rho_{\gamma_\xi(t)}dt \\
&= c\int_V a(v)dm_x^r(v)
\end{aligned}$$

where $c = e^{-2\delta^{-1}C(\delta^{-1})}$ and $a(v) = \int_r^{r+2\delta^{-1}} \rho_{\gamma_\xi(t)}dt$ with $\xi = r^{-1}\text{Exp}_x^{-1}v$. Suppose that $a(v) \geq -2\delta^{-1}k^2(n-1)$, then taking

$$U = \{v \in V : a(v) \leq -\frac{1}{2}\delta^2 c^{-1}(\Psi(\delta^{-1}))^{-1}\}$$

one obtains

$$\begin{aligned}
-\delta^2 c^{-1} &= \int_V a(v)dm_x^r(v) \\
&\geq -\frac{1}{2}\delta^2 c^{-1}(\Psi(\delta^{-1}))^{-1} m_x^r(V \setminus U) - 2\delta^{-1}k^2(n-1)m_x^r(U),
\end{aligned}$$

and so, by (2.17),

$$m_x^r(U) \geq (4k^2(n-1)c)^{-1}\delta^3,$$

implying the assumption of Lemma 2.11. ∎

The following result is a crucial consequence of the K-condition.

2.14. LEMMA: *Let M be a generalized n-dimensional CH-manifold satisfying the K-condition with the Ricci curvature bounded from below by $-k^2(n-1)$. Then one can pick up $L = L(K,k) > 0$ such that, for any $x, y \in M$, there exists a Borel set $G \subset B_y(L)$ satisfying*

$$m(G) \geq L^{-1} \quad \text{and} \quad Q_x(r,\xi) \geq L^{-1} \quad \textit{for all } (r,\xi) \in G, \tag{2.42}$$

where m is the Lebesque measure generated by the Riemannian volume on M.

Proof: Let $x, y \in M$. If $d(x,y) \geq K+1$, then take $r, s, U \subset S_x(r)$ as in the K-condition for the pair x, y. Since $U \subset B_y(K)$ then, necessarily, $r \geq 1$. If $d(x,y) < K+1$, then pick up $\tilde{y} \in S_x(K+1)$ and take $r, s, U \subset S_x(r)$ as in the K-condition for the pair $x, \tilde{y}$. Then, again $r \geq 1$, and since $B_y(3K+2) \supset B_{\tilde{y}}(K)$ it follows that in both cases

$$B_y(4K+2) \subset \tilde{G} = \bigcup_{r \leq t \leq r+s} \pi_x^t(U). \tag{2.43}$$

Set $\Xi = r^{-1}\mathrm{Exp}_x^{-1}U \subset S_xM$. Then by (2.4) and (2.22),

$$\begin{aligned} m_x^{r+s}(\pi_x^{r+s}(U)) &= \int_\Xi A_x(r+s,\xi)d\xi \\ &= \int_\Xi (\exp\int_r^{r+s} Q_x(\rho,\xi)d\rho)A_x(r,\xi)d\xi \\ &= \int_U (\exp\int_r^{r+s} Q_x(\rho,\xi(z))d\rho)dm_x^r(z) \\ &\geq (1+K^{-1})m_x^r(U) \end{aligned} \tag{2.44}$$

where $\xi(z) = r^{-1}\mathrm{Exp}_x^{-1}z$, $z \in U \subset S_x(r)$. Set

$$V_+ = \{z \in S_x(r) : \int_r^{r+s} Q_x(\rho,\xi(z))d\rho \geq \log(1+\frac{1}{2}K^{-1})\} \text{ and } V_- = S_x(r) \setminus V_+.$$

Since $r \geq 1$, then by (2.5) and (2.6) for all $\rho \geq r$,

$$0 < Q_x(\rho,\xi) \leq C = k\coth(k), \tag{2.45}$$

and so

$$\int_U \exp\left(\int_r^{r+s} Q_x(\rho,\xi(z))d\rho\right) dm_x^r(z) \tag{2.46}$$

$$\leq e^{Cs} m_x^r(U\cap V_+) + (1+\frac{1}{2}K^{-1})m_x^r(U\cap V_-).$$

By (2.44)–(2.46),

$$e^{Cs}\geq 1+K^{-1}, \quad \text{i.e.} \quad s\geq C^{-1}\log(1+K^{-1}) \tag{2.47}$$

and

$$m_x^r(U\cap V_+)\geq \frac{1}{2}K^{-1}e^{-Cs}m_x^r(U). \tag{2.48}$$

For $z\in V_+$ set $T_+(z)=\{t\in[r,r+s]: Q_x(t,\xi(z))\geq \frac{1}{2s}\log(1+\frac{1}{2}K^{-1})\}$ and $T_-(z)=[r,r+s]\setminus T_+(z)$. Then in the same way as above by (2.45), (2.47) and the definition of V_+ for any $z\in V_+$,

$$\operatorname{mes}(T_+(z))\geq \frac{1}{2}C^{-1}\log(1+\frac{1}{2}K^{-1}) \tag{2.49}$$

where mes denotes the Lebesgue measure on real numbers. Set

$$G=\left\{(\rho,\xi)\in\tilde{G}: Q_x(\rho,\xi)\geq \frac{1}{2K}\log\left(1+\frac{1}{2}K^{-1}\right)\right\}.$$

Since $s\leq K$ and $m_x^r(U)\geq K^{-1}$ then, by (2.47)–(2.49),

$$m(G)\geq \frac{1}{4}C^{-1}K^{-2}e^{-CK}\log(1+\frac{1}{2}K^{-1})\stackrel{\text{def}}{=}\ell. \tag{2.50}$$

This yields Lemma 2.14 with $L=\max(3K+2,\ell^{-1},KC\ell^{-1})$. ∎

2.15. PROPOSITION: *Let M be a generalized CH-manifold with the Ricci curvature bounded from below and $v(r)$ be as in (1.3). If*

$$\liminf_{t\to\infty} v(t)(t\Psi(t))^{-1}=\infty \tag{2.51}$$

where $\Psi(t)$ satisfies the right hand side of (2.17), then M satisfies the K-condition.

Proof: Pick up $x,y\in M$ arbitrarily. Fix $L\geq 1$ large enough; it will be specified in the proof. If $d(x,y)\geq L$ put $z=y$ and, if $d(x,y)<L$, choose $z\in S_x(L)$ so

that y belongs to the geodesic segment γ_{xz}. Let $r = d(x,z)$ and $\zeta = r^{-1}\mathrm{Exp}_x^{-1} z$. Set $U = B_x^r(z, L)$; then by (2.14) and the triangle inequality

$$V = \bigcup_{r+2L \geq t \geq r} \pi_x^t U \supset B_v(L)$$

where $v = \mathrm{Exp}_x(r+L)\zeta$. It follows from (2.4) and (2.5) that

$$m_x^{r+2L}(\pi_x^{r+2L} U) \geq m_x^t(\pi_x^t U)$$

for any $t \leq r + 2L$, and so

$$2Lm_x^{r+2L}(\pi_x^{r+2L}U) \geq m(V) \geq m(B_v(L)) \geq v(L) \geq v(L)(\Psi(L))^{-1} m_x^r(U).$$

Choose $L = \sup\{t \geq 1 : v(t)(t\Psi(t))^{-1} \leq 4\}$; then the K-condition will be satisfied with $K = 2L$. ∎

2.16. Proposition: *Suppose that M is a generalized CH-manifold (in fact, any complete noncompact Riemannian manifold) and, for some $N \geq 0$,*

$$\liminf_{r \to \infty} r^{-N} v(r) < \infty. \tag{2.52}$$

Then the top λ of the L^2-spectrum of Δ is equal to zero.

Proof: I employ the inequality (3.1) from [CY] which gives

$$0 \leq -\lambda \leq \frac{m(B_x(r))}{(r-\rho)^2 m(B_x(\rho))} \tag{2.53}$$

for any $x \in M$ and $r > \rho > 0$. Set

$$\alpha = \sup\left\{\beta : \liminf_{r\to\infty} r^{\beta} v(r) < \infty\right\}.$$

By (2.52), $\alpha \geq -N$, and so for any $\varepsilon > 0$,

$$\liminf_{r\to\infty} r^{\alpha-\varepsilon} v(r) = 0 \quad \text{and} \quad \liminf_{r\to\infty} r^{\alpha+\varepsilon} v(r) = \infty = \lim_{r\to\infty} r^{\alpha+\varepsilon} v(r). \tag{2.54}$$

Then there exist sequences $x_k \in M$ and $r_k \to \infty$ as $k \to \infty$ such that for all k large enough, $m(B_{x_k}(r_k)) \leq r_k^{-\alpha+\varepsilon}$. Taking in (2.53) $x = x_k$, $r = r_k$, and $\rho = \frac{1}{2} r_k$ one obtains by (2.53) and (2.54,) that, for $0 < \varepsilon \leq 1$,

$$0 \leq -\lambda \leq \frac{4m(B_{x_k}(r_k))}{r_k^2 m(B_{x_k}(1/2r_k))} \leq ((\tfrac{1}{2}r_k)^{\alpha+\varepsilon} v(\tfrac{1}{2}r_k))^{-1} 2^{2-\alpha-\varepsilon} r_k^{2\varepsilon-2} \to 0$$

as $k \to \infty$. ∎

2.17. COROLLARY: *Let M be a 2-dimensional generalized CH-manifold with the Ricci curvature bounded from below. Then the K-condition is equivalent to (1.3).*

Proof: The K-condition implies (1.3) by Proposition 2.9. If $\dim M = 2$ then $D_x^r(y, \rho)$ is 1-dimensional, and so $m_x^r(D_x^r(y, \rho)) \leq 2\rho$. Thus one can take $\Psi(t) = 2t$ in (2.17) which, together with (1.3), yields (2.51), and so the K-condition holds true. ∎

2.18. Proof of Assertion: If (1.3) is satisfied, then the K-condition holds true and we will see in the next section that this implies the positivity of $-\lambda$. On the other hand, if

$$\liminf_{r\to\infty} r^{-1}v(r) < \infty$$

then $\lambda = 0$ by Proposition 2.16. ∎

2.19. Remark: Propositions 2.15 and 2.16 say that if volumes of the disks $D_x^r(y, \rho)$ grow in ρ not faster than polynomially, then either volumes of the balls $B_x(r)$ grow faster than polynomially and then $\lambda < 0$, or they grow polynomially and then $\lambda = 0$. The class of generalized CH-manifolds for which volumes of the disks $D_x^r(y, \rho)$ grow not faster than polynomially in ρ includes symmetric spaces, but I do not know a general characterization of this class. In fact, in order to get $\lambda < 0$ one needs only the K-condition with one fixed pole x, and so it suffices to have that volumes of the disks $D_x^r(y, \rho)$ grow polynomially in ρ only for one fixed pole x. ∎

2.20. Remark: In fact, $\lambda < 0$ follows already if (3.2) below is satisfied for one fixed x. Thus in order to have $\lambda < 0$ it suffices to have the K-condition for one fixed x only and, correspondingly, the disks $D_x^r(y, \rho)$ should grow not faster than polynomially in ρ only for one fixed x (though the volumes of balls $B_x(r)$ should grow exponentially fast for all x). ∎

3. Heat kernel and the radial part of the Brownian motion

Recall that the Brownian motion $X(t)$ on M is a diffusion process generated by the Laplace–Beltrami operator Δ (see [IW] and [Ch]) and the heat kernel $p(t, x, y)$ is the transition density of $X(t)$, i.e. if $P(t, x, \Gamma)$, $\Gamma \subset M$ is the probability of the event $\{X(t) \in \Gamma\}$ provided $X(0) = x$, then

$$P(t, x, \Gamma) = \int_\Gamma p(t, x, y)dm(y) \leq 1. \tag{3.1}$$

Throughout this section I assume that M is a n-dimensional generalized CH-manifold satisfying the K-condition with the Ricci curvature bounded from below by $-k^2(n-1)$. Actually, I shall need here only the property expressed by Lemma 2.14 that Q_x is bounded away from zero on a "well spread" set. In the geodesic polar coordinates with a pole at x one can write $X(t) = (R_x(t), \Xi_x(t))$, where $\Xi_x(t) \in S_xM$ is the "angular" part of the Brownian motion $X(t)$ and $R_x(t) = d(x, X(t))$ is the "radial" part of $X(t)$, so that $X(t) = \mathrm{Exp}_x(R_x(t)\Xi_x(t))$.

3.1. PROPOSITION: *There exists $\alpha \in (0,1)$ such that for any $x, y \in M$ and all $t \geq 0$,*

$$P_y\{\alpha s \leq R_x(s) - R_x(0) \leq \alpha^{-1}s \text{ for all } s \geq t\} \geq 1 - \alpha^{-1}e^{-\alpha t}, \tag{3.2}$$

where $P_y\{\cdot\}$ denotes the probability of the event in brackets for the process $X(t)$ starting at y.

Proof: Since the Laplace–Beltrami operator Δ has the form (2.3) in geodesic polar coordinates, the radial part $R_x(t)$ of $X(t)$ satisfies the following stochastic equation:

$$R_x(t) = R_x(0) + \sqrt{2}w(t) + \int_0^t Q_x(X(s))ds, \tag{3.3}$$

where $w(t)$ is the one-dimensional Brownian motion (called the Wiener process) starting at zero. Let $L = L(K,k) > 0$ and $G = G_{x,y} \subset B_y(L)$ be the constant and the set satisfying (2.42). By the heat kernel comparison theorems (see [IW] or [Ch]), $p(t,y,z)$ is not less than the heat kernel $q_k(t,v,w)$ on the n-dimensional space of constant curvature $-k^2$ where v, w are any points with the distance between them equal to $d(y,z)$. It follows that there exists $b = b(k,L) > 0$ such that, for any $y \in M$,

$$\inf_{\frac{1}{2}\leq t\leq 1}\ \inf_{z\in B_y(L)} p(t,y,z) \geq b. \tag{3.4}$$

Set $\Psi_{x,y} = \int_0^1 \chi_{G_{x,y}}(X(t))dt$ where $\chi_G(x) = 1$ if $x \in G$ and $\chi_G(x) = 0$ if $x \notin G$. Then by (2.42) and (3.4),

$$E_y\Psi_{x,y} = \int_0^1 P_y\{X(t) \in G\}dt \geq \int_{\frac{1}{2}}^1 dt \int_G p(t,y,z)dm(z) \geq \frac{1}{2}bL^{-1}, \tag{3.5}$$

where E_y is the expectation for the process $X(t)$ provided $X(0) = y$. Introduce the events $A^+_{x,y} = \{\Psi_{x,y} \geq \frac{1}{4}bL^{-1}\}$ and $A^-_{x,y} = \Omega \setminus A^+_{x,y}$ where Ω is the path space of the Brownian motion, $X(t,\omega) = \omega(t)$, $\omega : [0,\infty) \to M$ is continuous. Since $\Psi_{x,y} \leq 1$, then by (3.5) in the same way as in (2.48),

$$P_y\{A^+_{x,y}\} \geq \frac{1}{4}bL^{-1}. \tag{3.6}$$

Let $\theta^s : \Omega \to \Omega$ be the shift operator so that $(\theta^s\omega)(t) = \omega(t+s)$. Then $\theta^s A^+_{x,X(0)} = \{\theta^s\Psi_{x,X(0)} \geq \frac{1}{4}bL^{-1}\}$ where

$$\theta^s\Psi_{x,X(0)} = \int_0^1 \chi_{G_{x,X(s)}}(X(t+s))dt.$$

By (2.5), (2.42) and (3.3)

$$\begin{aligned} R_x(t) &\geq R_x(0) + \sqrt{2}w(t) + \sum_{1\leq j\leq t} \int_{j-1}^{j} Q_x(X(s))ds \\ &\geq R_x(0) + \sqrt{2}w(t) + L^{-1} \sum_{0\leq j\leq t-1} \theta^j\Psi_{x,X(0)} \\ &\geq R_x(0) + \sqrt{2}w(t) + \frac{1}{4}bL^{-2}N_t \end{aligned} \tag{3.7}$$

where $N_t = \sum_{0\leq j\leq t-1} \chi_{\theta^j A^+_{x,X(0)}}$ and $\chi_A = 1$ if the event A occurs and $\chi_A = 0$, otherwise. Let $\mathcal{E}^t_{j_1,\dots,j_\ell}$, $0 \leq j_1 < j_2 \cdots < j_\ell \leq t-1$ be the event that all $\theta^{j_i}A^+_{x,X(0)}$, $i = 1,\dots,\ell$ occur and, if $j \neq j_i$, $i = 1,\dots,\ell$ and $0 \leq j \leq t-1$, then $\theta^j A^-_{x,X(0)}$ occurs, i.e.

$$\mathcal{E}^t_{j_1,\dots,j_\ell} = \left(\bigcap_{0\leq i\leq \ell} \theta^{j_i}A^+_{x,X(0)}\right) \cap \left(\bigcap_{j\neq j_i} \theta^j A^-_{x,X(0)}\right).$$

By (3.6) and the Markov property (see [IW] or [KS]) of the Brownian motion $X(t)$,

$$\begin{aligned} P_y\{\mathcal{E}^t_{j_1,\dots,j_\ell}\} &\leq E_y\chi_{A_1}E_{X(1)}\chi_{\theta^1 A_2}\cdots E_{X([t-1])}\chi_{\theta^{[t-1]}A_{[t-1]}} \\ &\leq (1-\frac{1}{4}bL^{-1})^{[t]-\ell-1}, \end{aligned} \tag{3.8}$$

where $[\cdot]$ denotes the integral part, $A_{j_i} = A^+_{x,X(0)}$ for all $i = 1,\dots,\ell$ and $A_j = A^-_{x,X(0)}$ if $j \neq j_i$ for any i. Since by (2.42), (3.1) and (3.4), $bL^{-1} \leq 1$, the above inequality makes sense. Thus by the Stirling formula

$$
\begin{aligned}
P_y\{N_t \le \beta t\} &= \sum_{0\le \ell \le \beta t}\ \sum_{0\le j_1<j_2\dots<j_\ell \le t-1} \mathcal{E}^t_{j_1,\dots,j_\ell} \\
&\le \sum_{0\le \ell\le \beta} \binom{[t]-1}{\ell}\left(1-\frac{1}{4}bL^{-1}\right)^{[t]-\ell-1} \\
&\le \beta t \frac{(t-1)^{\beta t}}{[\beta t]!}\left(1-\frac{1}{4}bL^{-1}\right)^{t(1-\beta)-2} \\
&\le t^2(6\beta^{-1})^{\beta t}\left(1-\frac{1}{4}bL^{-1}\right)^{t(1-\beta)-2}
\end{aligned} \tag{3.9}
$$

provided $0 < \beta \le \frac{1}{3}$ and $t \ge 2\beta^{-1}$. Therefore if β is so small that $(6\beta^{-1})^\beta(1-\frac{1}{4}bL^{-1})^{1-\beta} \le (1-\frac{1}{5}bL^{-1})$, then one can choose $\tilde{C} > 0$ such that

$$
P_y\{N_t \le \beta t\} \le \tilde{C}(1-\frac{1}{6}bL^{-1})^t. \tag{3.10}
$$

Now by (3.7) and (3.10),

(3.11)

$$
\begin{aligned}
&P_y\{R_x(s)-R_x(0) \le \tfrac{1}{8}bL^{-2}\beta s \text{ for some } s \ge t\} \\
&\le P_y\{\inf_{m<s\le m+1}(R_x(s)-R_x(0)) \le \tfrac{1}{8}bL^{-2}\beta(m+1) \text{ for some } m > t-1\} \\
&\le \sum_{m>t-1} P_y\{\tfrac{1}{4}bL^{-2}N_m + \sqrt{2}\inf_{m<s\le m+1} w(s) \le \tfrac{1}{8}bL^{-2}\beta(m+1)\} \\
&\le \sum_{m>t-1} (P_y\{N_m \le \beta m\} + P\{\inf_{0\le s\le m+1} w(s) \le -\tfrac{1}{12}bL^{-2}\beta(m-1)\}).
\end{aligned}
$$

Since $w(s)$ and $-w(s)$ have the same distribution then by the reflection principle (see [KS]) for any $u, \rho > 0$,

$$
\begin{aligned}
P\{\inf_{0\le s\le u} w(s) \le -\rho\} &= P\{\sup_{0\le s\le u} w(s) \ge \rho\} = 2P\{w(u)\ge \rho\} \\
&= \sqrt{\frac{2}{\pi}}\int_\rho^\infty e^{-v^2/2u}dv \le \frac{2u}{\rho}e^{-\rho^2/2u}.
\end{aligned} \tag{3.12}
$$

Now from (3.10)–(3.12) it follows that for some $\alpha > 0$,

$$
P_y\{R_x(s)-R_x(0) \le \alpha s \text{ for some } s \ge t\} \le \frac{1}{2}\alpha^{-1}e^{-\alpha t}. \tag{3.13}
$$

Set $\tau_x(r) = \inf\{t \ge 0 : X(t) \in S_x(r)\}$, i.e. $\tau_x(r)$ is the first hitting time by $X(t)$ of the geodesic sphere $S_x(r)$. To complete the proof of Proposition 3.1, I need the following result which will be employed also in Section 4.

3.2. LEMMA: *Let $\infty \geq r_3 > r_2 > r_1 > 0$, $x \in M$, $\xi \in S_xM$, and $y = \mathrm{Exp}_x(r_2\xi)$. Then*

$$e^{-C(r_1)(r_2-r_1)} - (e^{C(r_1)(r_3-r_1)} - 1)^{-1} \leq P_y\{\tau_x(r_3) > \tau_x(r_1)\} \leq 2\alpha^{-1}\exp(-\frac{\alpha}{2}(r_2 - r_1)) \tag{3.14}$$

where $C(r) = k\coth(kr)$ and $\alpha > 0$ satisfies (3.13). Furthermore, for $t \geq \alpha^{-1}(r_3 - r_2)$,

$$P_y\{\tau_x(r_3) \geq t\} \leq \alpha^{-1}e^{-\alpha t}, \tag{3.15}$$

where $\alpha > 0$ satisfies (3.13).

Proof: Since $R_x(0) = r_2$, then by (2.5) and (3.3), $R_x(t) - r_2 \geq \sqrt{2}w(t)$, and so by the reflection principle (3.12),

$$\begin{aligned} P_y\{\tau_x(r_1) \leq \tfrac{1}{2}(r_2 - r_1)\} &= P_y\{\inf_{0\leq s\leq \frac{1}{2}(r_2-r_1)}(R_x(s) - r_2) \leq r_1 - r_2\} \\ &\leq P\{\inf_{s\leq \frac{1}{2}(r_2-r_1)} w(s) \leq \tfrac{1}{\sqrt{2}}(r_1 - r_2)\} \leq \sqrt{2}e^{-\frac{1}{2}(r_2-r_1)}. \end{aligned} \tag{3.16}$$

On the other hand, by (3.13),

$$R_x(t) - R_x(0) \geq \frac{\alpha}{2}(r_2 - r_1) \quad \text{for all } t \geq \frac{1}{2}(r_2 - r_1),$$

with P_y-probability of at least $1 - \alpha^{-1}\exp(-\frac{\alpha}{2}(r_2 - r_1))$, and so

$$P_y\left\{\infty > \tau_x(r_1) \geq \frac{1}{2}(r_2 - r_1)\right\} \leq \alpha^{-1}\exp\left(-\frac{\alpha}{2}(r_2 - r_1)\right), \tag{3.17}$$

which, together with (3.16), yields the right hand side of (3.14). The left hand side of (3.14), which I will need only in Section 5, follows in the same way as in Lemma 3.1 from [Ki] by the comparison

$$R_x(t) - R_x(0) \leq \sqrt{2}w(t) + C(r_1)t \quad \text{if } t \leq \tau_x(r_1) \text{ and } R_x(0) \geq r_1 \tag{3.18}$$

satisfied in view of (2.6) and (3.3). Next, by (3.13) for $\alpha t \geq r_3 - r_2$,

$$\begin{aligned} P_y\{\tau(r_3) \geq t\} &\leq P_y\{R_x(t) - R_x(0) < r_3 - r_2\} \\ &\leq P_y\{R_x(t) - R_x(0) < \alpha t\} \leq \alpha^{-1}e^{-\alpha t}, \end{aligned} \tag{3.19}$$

proving (3.15). ∎

Next, I am able to conclude the proof of Proposition 3.1. By (3.12) and (3.18), for any $r_1, s > 0$, $r_2 > r_1$, $r_3 > r_2 + C(r_1)s$, and $z \in S_x(r_2)$,

$$\begin{aligned} P_z\{\tau(r_3) \le s \wedge \tau_x(r_1)\} &\le P\{\sqrt{2} \sup_{0 \le u \le s} w(u) \ge r_3 - r_2 - C(r_1)s\} \\ &\le 2\sqrt{2s}\exp(-(r_3 - r_2 - C(r_1)s)^2/4s), \end{aligned} \tag{3.20}$$

where $a \wedge b = \min(a, b)$.

Set $d = d(x, y)$ and $C = C(1)$; then $\tau_x(C\ell + d) < \tau_x(3C\ell + d)$ if $X(0) = y$, and so by the strong Markov property (see [IW] of [KS]) of the process $X(t)$,

$$\begin{aligned} &P_y\{R_x(s) - R_x(0) \ge 3Cs \text{ for some } s \ge t\} \\ &\le \sum_{\ell \ge t-1} P_y\{\sup_{\ell \le s \le \ell+1} R_x(s) \ge 3C\ell + d\} \\ &\le \sum_{\ell \ge t-1} P_y\{\tau_x(3C\ell + d) \le \ell + 1\} \\ &\le \sum_{\ell \ge t-1} E_y P_{X(\tau_x(C\ell+d))}\{\tau_x(3C\ell + d) \le \ell + 1\} \\ &\le \sum_{\ell \ge t-1} \sup_{z \in S_x(C\ell+d)} P_z\{\tau_x(3C\ell + d) \le \ell + 1\}. \end{aligned} \tag{3.21}$$

To estimate the last probability in (3.21) I employ the inequality

$$P_z\{\tau(r_3) \le s\} \le P_z\{\tau(r_3) \le s \wedge \tau_x(r_1)\} + P_z\{\tau(r_3) \ge \tau_x(r_1)\} \tag{3.22}$$

with $r_1 = 1$, $r_2 = C\ell + d$, $r_3 = 3C\ell + d$, $z \in S_x(r_2)$, and $s = \ell + 1$. Next, I estimate the first probability in the right hand side of (3.22) by (3.20) and the second probability there by (3.14), which together with (3.21) yields

$$P_y\{R_x(s) - R_x(0) \ge \alpha^{-1}s \text{ for some } s \ge t\} \le \frac{1}{2}\alpha^{-1}e^{-\alpha t} \tag{3.23}$$

provided α is small enough. Finally, (3.13) and (3.23) imply (3.2). ∎

3.3. Corollary: *For any $x, y \in M$ with P_y-probability one,*

$$\alpha \le \liminf_{t\to\infty} \frac{1}{t} d(x, X(t)) \le \limsup_{t\to\infty} \frac{1}{t} d(x, X(t)) \le \alpha^{-1}. \tag{3.24}$$

Proof: The assertion follows from (3.2) and the Borel–Cantelli lemma. ∎

3.4. Proof of Theorem A: I employ upper bounds for the heat kernel from [CGT], p. 27 which in our circumstances amount to the following estimates:

$$(3.25)\quad \begin{aligned} p(t,x,y) &\le C(t,\delta) \\ &= \tilde{C}\max(t^{-n/2}+\delta^{-n},(t^{1-n/2}+2\delta^{-(n+2)})(t^{-1-n/2}+\delta^{-(n+2)})) \end{aligned}$$

for all $\delta, t > 0$, any $x, y \in M$, and some $\tilde{C}$ independent of δ, t, x. If $d(x,y) \ge 2\delta$, then one has a better estimate:

$$(3.26)\qquad p(t,x,y) \le C(t,\delta)\exp(-\tfrac{1}{4}(d(x,y)-2\delta)^2t^{-1}).$$

Next, if $d(x,y)+\alpha t/4 \ge 2$, then by (3.2), (3.25), (3.26) and by the semigroup property of the heat kernel

(3.27)

$$\begin{aligned} p(t,x,y) =& \int_M p\left(\frac{t}{2},x,z\right)p\left(\frac{t}{2},z,y\right)dm(z) \\ =& \int_{B_y(d(x,y)+\alpha t/4)} p\left(\frac{t}{2},x,z\right)p\left(\frac{t}{2},z,y\right)dm(z) \\ &+ \int_{M\setminus B_y(d(x,y)+\alpha t/4)} p\left(\frac{t}{2},x,z\right)p\left(\frac{t}{2},z,y\right)dm(z) \\ \le& C\left(\frac{t}{2},1\right)P_x\left\{R_y\left(\frac{t}{2}\right)\le d(x,y)+\alpha t/4\right\} \\ &+ \sup_{z:d(z,y)\ge d(x,y)+\alpha t/4} p\left(\frac{t}{2},z,y\right) \\ \le& C\left(\frac{t}{2},1\right)\alpha^{-1}e^{-\alpha t/2}+C\left(\frac{t}{2},1\right)\exp(-\tfrac{1}{4}(d(x,y)+\alpha t/4-2)^2t^{-1}) \end{aligned}$$

taking into account that $R_y(0) = d(x,y)$ provided $X(0) = x$. Finally, (1.2) follows from (3.25)–(3.27) provided $d(x,y)+\alpha t/4 \ge 2$ and (1.2) follows from (3.25) if $0 \le d(x,y)+at/4 \le 2$, $t > 0$. Since

$$(3.28)\qquad e^{t\Delta}f(x) = \int_M p(t,x,y)f(t)dm(y),$$

then by (1.2) the spectral radius of the operator e^{Δ} does not exceed e^{-c}. Thus the supremum of the spectrum of the self-adjoint extension of Δ to $L^2(M,m)$ does not exceed $-c$. ∎

3.5. Remark: Looking carefully at the proofs in Sections 1 and 2 one can give a positive lower bound for the constant C in (1.2) in terms of three geometric quantities: the dimension n, the lower bound on the Ricci curvature $-k^2(n-1)$, and either K from the K-condition or one of parameters characterizing hyperbolicity of M (such as C in C-exponentially fast divergence of geodesics, δ, if all geodesic triangles are δ-thin, etc.). ∎

3.6. LEMMA: *There exists $\beta > 0$ such that, for any $x \in M$ and all $t > 0$,*

$$P_x\{\tau_x(1) \le t\} \le \beta^{-1} \exp(-\beta t^{-1}). \tag{3.29}$$

Proof: Set $\tilde{C} = C(1/2)$ then by (3.12), (2.19) and the strong Markov property of the process $X(t)$,

$$\begin{aligned} P_x\{\tau_x(1) \le t\} &\le E_x P_{X(\tau_x(3/4))}\{\tau_x(1/2) \wedge \tau_x(1) \le t\} \\ &\le \sup_{z \in S_x(3/4)} P_z\{\tau_x(1/2) \wedge \tau_x(1) \le t\} \\ &\le P\{\sqrt{2} \sup_{0 \le s \le t} w(s) \ge \frac{1}{4} - \tilde{C}t\} \\ &\le 2\sqrt{2}t(\frac{1}{4} - \tilde{C}t)^{-1} \exp(-(\frac{1}{4} - \tilde{C}t)^2/4t) \end{aligned} \tag{3.30}$$

provided $t < (4\tilde{C})^{-1}$. This gives (3.29) if β is chosen small enough. ∎

4. Harmonic measures and the "angular" part of the Brownian motion

Throughout this and the next section I assume that M is a hyperbolic n-dimensional generalized CH-manifold with the Ricci curvature bounded from below by $-k^2(n-1)$.

4.1. LEMMA (cf. [KL] and [BK]): *There exists $c > 0$ such that, if $\infty > r_3 > r_2 > r_1 \ge r_0 > 1$, $x \in M$, $y \in S_x(r_2)$, and*

(4.1)

$$t_x = t_x(\rho; r_0, r_1) = \inf\{d_x^{r_1 - 1}(z, v) : z, v \in S_x(r_1) \text{ and } d_x^{r_0}(\pi_x^{r_0}(z), \pi_x^{r_0}(v)) \ge \rho\}^{1/2}$$

then

$$\begin{aligned} Q_y(x; r_0, r_1, r_3; \rho) &\overset{\text{def}}{=} P_y\{\sup_{0 \le s \le \tau_x(r_1) \wedge \tau_x(r_3)} d_x^{r_0}(\pi_x^{r_0} X(s), \pi_x^{r_0} y) \ge \rho\} \\ &\le P_y\{\tau_x(r_1) \wedge \tau_x(r_3) > t_x\} + c^{-1} t_x^2 e^{-ct_x} \end{aligned} \tag{4.2}$$

for any $\rho > 0$ such that $t_x \geq 1$.

Proof: Clearly,

$$Q_y(x; r_0, r_1, r_3; \rho) \leq P_y\{\tau_x(r_1) \wedge \tau_x(r_3) > t_x\} + \tilde{Q}_y, \tag{4.3}$$

where

$$\tilde{Q}_y = P_y \left\{ \sup_{0 \leq t \leq \tau_x(r_1) \wedge \tau_x(r_3)} d_x^{r_0}(\pi_x^{r_0} X(t), \pi_x^{r_0} y) \geq \rho \text{ and } \tau_x(r_1) \wedge \tau_x(r_3) \leq t_x \right\}.$$

The event in the last probability can occur only if the Brownian motion $X(t)$ will cover a distance of at least t_x^2 in time not exceeding t_x in the following sense. Let $\ell = [t_x^2]$ and define the random points y_i, $i = 0, \ldots, \ell - 1$ inductively $y_0 = y$, $y_{i+1} = X(\tau_{y_i}(1))$. Then

$$t_x \geq \sum_{\ell - 1 \geq i \geq 0} \tau_{y_i}(1). \tag{4.4}$$

Hence $\tau_{y_i}(1) \leq t_x \ell^{-1}$ for some $i = 0, \ldots, \ell - 1$ which, together with the strong Markov property, enables one to conclude that

$$\tilde{Q}_y \leq \ell \sup_{z \in M} P_z\{\tau_z(1) \leq t_x \ell^{-1}\}. \tag{4.5}$$

Finally, (4.2) follows from (3.29), (4.3) and (4.5). ∎

The following result is the key step in the proof of Theorem B (cf. Lemma 3.1 in [KL] and Proposition 4.3 in [BK]).

4.2. Proposition: *There exists $a > 0$ such that for any $x \in M$, all $r_2 \geq r_1 \geq \rho \geq a^{-1}$, and each $z \in S_x(r_2)$,*

$$P_z \left\{ \sup_{t \geq 0} d_x^{r_1}(\pi_x^{r_1} X(t), \pi_x^{r_1} z) \leq \rho \right\} \geq 1 - a^{-1} \rho^{-a} e^{-a(r_2 - r_1)}. \tag{4.6}$$

Proof: First, I derive (4.6) for $r_1 = r_2$. Since M is a hyperbolic manifold there exists $C > 0$ such that (2.2) holds true. Assume that $a^{-1} \geq 12C + 1$ so that $\rho \geq 12C + 1$, as well, and notice that if $y_1, y_2 \in S_x(r)$, $d_x^r(y_1, y_2) \geq 3C$ then (2.2) gives

$$d_x^{r+u}(\pi_x^{r+u} y_1, \pi_x^{r+u} y_2) > 2 d_x^r(y_1, y_2) \quad \text{for all } u \geq C^2 + 3C. \tag{4.7}$$

Set $r_\varepsilon(j) = r_1 + \varepsilon(j-1)\log\rho + \frac{1}{2}j(j-1)(C^2+3C)$ for all $j = 0,1,2,\dots$ where ε is chosen to satisfy

$$0 < \varepsilon \le \log\left(\rho^3/4^4\right)\left(4k(\log\rho)\coth k\right)^{-1}. \tag{4.8}$$

Define $\ell_1^\varepsilon(z) = \{y \in S_x(r_1) : d_x^{r_1}(y,z) \le \rho/4\}$, $\partial_1^-(z) = \pi_x^{r_\varepsilon(0)}\ell_1^\varepsilon(z)$, and $\partial_1^+(z) = \pi_x^{r_\varepsilon(2)}\ell_1^\varepsilon(z)$. Furthermore, for $j = 1,2,\dots$ I define by induction

$$\begin{aligned}\partial_{j+1}^-(z) &= \{y \in S_x(r_\varepsilon(j)) : d_x^{r_\varepsilon(j)}(y, \ell_j^\varepsilon(z)) \le 2^{j-3}\rho\},\\ \ell_{j+1}^\varepsilon(z) &= \pi_x^{r_\varepsilon(j+1)}\partial_{j+1}^-(z) \quad \text{and} \quad \partial_{j+1}^+(z) = \pi_x^{r_\varepsilon(j+2)}\partial_{j+1}^-(z).\end{aligned}$$

Set also for $j = 1,2,\dots,$

$$D_j^\varepsilon(z) = \bigcup_{r_\varepsilon(j-1)\le u\le r_\varepsilon(j+1)} \pi_x^u \partial_j^-(z)$$

and $\partial_j^s(z) = \partial D_j^\varepsilon(z) \setminus (\partial_j^+(z) \cup \partial_j^-(z))$, where ∂U denotes the boundary of a set U. The picture here is similar to Figure 3.1 from [KL] but parameters of the construction are different. We remark that in view of (4.7),

$$\pi_x^{r_1}\left(\bigcup_{j\ge 1} D_j^\varepsilon(z)\right) \subset \{y \in S_x(r_1) : d_x^{r_1}(y,z) \le \rho\}. \tag{4.9}$$

Let $\tau_j = \inf\{t \ge 0 : X(t) \notin D_j^\varepsilon(z)\}$ be the first exit time of the Brownian motion $X(t)$ from $D_j^\varepsilon(z)$. Then for any $y \in \partial_{j-1}^+(z)$ if $j > 1$ and for $y = z$ if $j = 1$, it follows that

$$\begin{aligned}P_y\{X(\tau_j) \notin \partial_j^+(z)\} \le & P_y\{\tau_x(r_\varepsilon(j+1)) > \tau_x(r_\varepsilon(j-1))\}\\ & + P_y\{X(\tau_j) \in \partial_j^s(z)\}.\end{aligned} \tag{4.10}$$

By (3.14),

$$\begin{aligned}&\sup_{y\in S_x(r_\varepsilon(j))} P_y\{\tau_x(r_\varepsilon(j+1)) > \tau_x(r_\varepsilon(j-1))\}\\ &\le 2\alpha^{-1}\rho^{-\alpha\varepsilon/2}\exp\left(-\frac{\alpha}{2}(j-1)(C^2+3C)\right).\end{aligned} \tag{4.11}$$

By (2.9), (2.14) and (4.8),

$$\partial_1^-(z) \supset \{y \in S_x(r_\varepsilon(0)) : d_x^{r_\varepsilon(0)}(y, \pi_x^{r_\varepsilon(0)}z) \le \rho^{1/4}\}, \tag{4.12}$$

and so

$$P_z\{X(\tau_1) \in \partial_1^s(z)\} \le Q_z(x; r_\varepsilon(0), r_\varepsilon(0), r_\varepsilon(2), \rho^{1/4}) \tag{4.13}$$

with Q_z defined by (4.2). Furthermore, by (2.14) and the definition of $D_j^\varepsilon(z)$ for any $y \in \partial_{j-1}^+$, $j = 2, 3, \ldots$,

$$P_y\{X(\tau_j) \in \partial_j^s(z)\} \le Q_y(x; r_\varepsilon(j-1), r_\varepsilon(j-1), r_\varepsilon(j+1), 2^{j-3}\rho). \tag{4.14}$$

Next, I remark that by (3.15) for any $y \in S_x(r_\varepsilon(n))$, $j = 1, 2, \ldots$,

$$\begin{aligned} &P_y\{\tau_x(r_\varepsilon(j-1)) \wedge \tau_x(r_\varepsilon(j+1)) \ge t_x^{(j)}\} \\ &\le P_y\{\tau_x(r_\varepsilon(j+1)) \ge t_x^{(j)}\} \le \alpha^{-1}\exp(-\alpha t_x^{(j)}) \end{aligned} \tag{4.15}$$

provided

$$t_x^{(j)} \ge \alpha^{-1}(\varepsilon \log \rho + j(C^2 + 3C)). \tag{4.16}$$

Taking $t_x^{(1)} = \rho^{1/8}$, $t_x^{(j)} = 2^{(j-3)/2}\rho^{1/2}$ for $j = 2, 3, \ldots$, and choosing $a > 0$ so small that (4.16) holds true for any $\rho \ge a^{-1}$, one derives by (4.1) and (4.10)–(4.15) that for all $j \ge 1$,

$$P_y\{X(\tau_j) \notin \partial_j^+(z)\} \le c_1^{-1}\rho^{-c_1}e^{-c_1 j} \tag{4.17}$$

for some constant $c_1 > 0$ independent of $z, y \in M$ provided $y = z$ if $j = 1$ and $y \in \partial_{j-1}^+(z)$ if $j > 1$.

By (4.9), (4.17) and the strong Markov property of the process $X(t)$ for $z \in S_x(r_1)$, it follows that

$$\begin{aligned} &P_z\{\sup_{t\ge 0} d_x^{r_1}(\pi_x^{r_1}X(t), z) \le \rho\} \\ &\ge E_z\chi_{X(\tau_1)\in\partial_1^+(z)}E_{X(\tau_1)}\chi_{X(\tau_2)\in\partial_2^+(z)}\cdots \\ &\ge 1 - P_z\{X(\tau_1) \notin \partial_1^+(z)\} - \sum_{j=2}^\infty \sup_{y\in\partial_{j-1}^+(z)} P_y\{X(\tau_j) \notin \partial_j^+(z)\} \\ &\ge 1 - c_1^{-2}\rho^{-c_1}, \end{aligned} \tag{4.18}$$

implying (4.6) for $r_1 = r_2$. Now let $r_2 > r_1$. Then employing (4.18) for r_2 in place of r_1 and $z \in S_x(r_2)$, one obtains by (2.2),

$$\begin{aligned} &P_z\{\sup_{t\ge 0} d_x^{r_1}(\pi_x^{r_1}X(t), \pi_x^{r_1}z) \le \rho\} \\ &\ge P_z\left\{\sup_{t\ge 0} d_x^{r_2}(\pi_x^{r_2}X(t), z) \le \left(\frac{3}{4}\right)^C \left(\frac{4}{3}\right)^{C^{-1}(r_2-r_1)} \rho\right\} \\ &\ge 1 - c_1^{-2}\rho^{-c_1}\left(\frac{4}{3}\right)^{c_1C}\left(\frac{3}{4}\right)^{c_1C^{-1}(r_2-r_1)}, \end{aligned} \tag{4.19}$$

yielding (4.6) for a small enough. ∎

4.3. COROLLARY: *For any $\delta > 0$, $\delta \le 2$ set*

$$r_x(\delta) = \max(a^{-1}, q_x^{-1}((2c_x a \arcsin(\delta/2))^{-1}))$$

where c_x, q_x are the same as in Lemma 2.3, q_x^{-1} is the inverse to the q_x function, and a is the same as in Proposition 4.2. Then for any $r \ge r_x(\delta)$ and $z = \mathrm{Exp}_x(r\xi)$, $\xi \in S_xM$,

$$P_z\left\{\sup_{t\ge 0} \|\Xi_x(t) - \xi\| > \delta\right\} \le a^{-1}a^a e^{-a(r-r_x(\delta))}. \tag{4.20}$$

Proof: If $\xi, \eta \in S_xM$ then, by (2.13),

$$\begin{aligned} d_x^{r_1}(\mathrm{Exp}_x(r_1\xi), \mathrm{Exp}_x(r_1\eta)) &\ge c_x q_x(r_1)\angle_x(\xi,\eta) \\ &= 2c_x q_x(r_1)\arcsin(\|\xi-\eta\|/2). \end{aligned} \tag{4.21}$$

Set $r_1 = r_x(\delta)$. Since q_x is an increasing function then, by (4.21),

$$d_x^{r_1}(\mathrm{Exp}_x(r_1\xi), \mathrm{Exp}_x(r_1\eta)) > a^{-1} \quad \text{provided} \quad \|\xi - \eta\| > \delta.$$

Thus by (4.6) with $\rho = a^{-1}$ and $r_2 = r$,

$$\begin{aligned} P_z\left\{\sup_{t\ge 0}\|\Xi_x(t)-\xi\| > \delta\right\} &\le P_z\left\{\sup_{t\ge 0} d_x^{r_1}(\pi_x^{r_1}X(t), \pi_x^{r_1}z) > a^{-1}\right\} \\ &\le a^{-1}a^a e^{-a(r-r_x(\delta))}. \end{aligned}$$ ∎

4.4. COROLLARY: *For any $x, y \in M$ with P_y-probability one, the limits*

$$\lim_{t\to\infty}\Xi_x(t) = \lim_{r\to\infty}\Xi_x(\tau_x(r)) = \Xi_x(\infty) \in S_xM \tag{4.22}$$

exist and are the same.

Proof: By (4.20) and the strong Markov property of $X(t)$,

$$\begin{aligned} &P_y\left\{\sup_{s\ge\tau_x(r)} \|\Xi_x(s) - \Xi_x(\tau(r))\| > \delta\right\} \\ &\le E_y P_{X(\tau_x(r))}\left\{\sup_{s\ge 0}\|\Xi_x(s) - \Xi_x(0)\| > \delta\right\} \\ &\le \sup_{z\in S_x(r)} P_z\left\{\sup_{s\ge 0}\|\Xi_x(s) - r^{-1}\mathrm{Exp}_x^{-1}z\| > \delta\right\} \\ &\le a^{-1}a^a e^{-a(r-r_x(\delta))}. \end{aligned} \tag{4.23}$$

Taking $\delta = \delta_j = j^{-1}$ and $r = r_j = a^{-1}(j + r_x(j^{-1}))$, one derives by the Borel–Cantelli lemma that there exists a random number $\ell = \ell(\omega)$ such that $\ell < \infty$ P_y-almost surely (a.s.) and

$$(4.24) \qquad \sup_{s \geq \tau_x(r_j)} ||\Xi_x(s) - \Xi(\tau_x(r_j))|| \leq j^{-1} \quad \text{for all } j \geq \ell,$$

implying the convergence with probability one of $\Xi_x(\tau_x(r))$ as $r \to \infty$ to a random vector $\Xi_x(\infty) \in S_xM$. Next, by (3.15),

$$P_y\{\tau_x(r) < \alpha^{-1}(r-d)\} \geq 1 - \alpha^{-1}e^{-(r-d)}$$

where $d = d(x,y)$, and so by (4.23) for any $t \geq \alpha^{-1}(r-d)$,

$$(4.25) \qquad P_y\{||\Xi_x(t) - \Xi_x(\tau_x(r))|| > \delta\} \leq \alpha^{-1}e^{-(r-d)} + a^{-1}a^a e^{-a(r-r(\delta))}.$$

Choosing again $\delta = \delta_j = j^{-1}$ and $r = r_j = a^{-1}(j + r_x(j^{-1}))$, we conclude by the Borel–Cantelli lemma that there exists a random number $\ell = \ell(\omega)$ so that $\ell < \infty$ P_y-a.s. and

$$||\Xi_x(t) - \Xi_x(\tau_x(r_j))|| \leq j^{-1}$$

provided $t \geq \alpha^{-1}(r_j - d)$ and $j \geq \ell$. This yields the convergence with probability one of $\Xi_x(t)$ as $t \to \infty$ to the same random vector $\Xi_x(\infty)$. ∎

Remark that Corollary 3.3 together with Corollary 4.4 yield that with P_y-probability one, $X(t)$ converges as $t \to \infty$ in the cone topology on $M \cup S(\infty)$ (see [Go2]) to a random point $X(\infty) \in S(\infty)$, so that $X(\infty) = \Phi_x \Xi_x(\infty)$ where $\Phi_x : S_xM \to S(\infty)$ was defined in Introduction. Next, I define the harmonic measures on $S(\infty)$ for any Borel $\Gamma \subset S(\infty)$ by

$$(4.26) \qquad P(x,\Gamma) = P_x\{X(\infty) \in \Gamma\} = P_x\{\Phi_z \Xi_z(\infty) \in \Gamma\}$$

which, clearly, is independent of $z \in M$.

4.5. COROLLARY: *Harmonic measures $P(x;\cdot)$ are positive on open subsets of $S(\infty)$.*

Proof: By the heat kernel comparison theorems (see [IW] or [Ch]) $p(t,x,z)$ is not less than the heat kernel on the n-dimensional space of constant curvature $-k^2$, in particular, $p(t,x,z) > 0$. Let Γ be an open subset of $S(\infty)$, $\zeta \in \Gamma$,

$\xi = \Phi_x^{-1}\zeta$, and $\Phi_x U \subset \Gamma$ where $U = \{\eta \in S_xM : ||\xi - \eta|| \leq 2\delta\}$. Take $r > 0$ so big that $a^{-1}a^a e^{-a(r-1-r_x(\delta))} \leq 1/2$ and $r \geq r_x(\delta) + 1$. Then by (4.20) and the Markov property of $X(t)$,

$$P(x,\Gamma) \geq \int_{B_z(1)} p(t,x,y) P_y\{\sup_{s\geq 0} ||\Xi_x(s) - \eta(y)|| \leq \delta\}$$
$$\geq \frac{1}{2} P_x\{X(t) \in B_z(1)\} > 0,$$

where $z = \mathrm{Exp}_x(r\xi)$ and $\eta(y) = (d(x,y))^{-1}\mathrm{Exp}_x^{-1} y$. ∎

4.6. COROLLARY: *The formula*

$$h_f(x) = \int_{S(\infty)} f(\zeta) P(x, d\zeta) \tag{4.27}$$

gives a one-to-one correspondence between continuous functions f on $S(\infty)$ and harmonic functions h_f having a continuous extension f to $S(\infty)$, i.e., (4.27) solves the Dirichlet problem at infinity.

Proof: Let P_t, $t \geq 0$ be the "heat" semigroup of operators given by

$$P_t g(x) = \int_M p(t,x,y) g(y) dm(y) = E_x g(X_t). \tag{4.28}$$

By (4.26) and the Markov property of $X(t)$ for any Borel $\Gamma \subset S(\infty)$,

$$E_x P(X(t), \Gamma) = P(x, \Gamma),$$

and so $P_t h_f = h_f$, which implies that any h_f given by (4.27) is harmonic. It follows from (4.20) that $P(x,\cdot)$ weakly converges as $x \to \zeta \in S(\infty)$ to the atomic measure concentrated at ζ. Thus

$$\lim_{x \to \zeta} h_f(x) = f(\zeta)$$

for any continuous function f on $S(\infty)$, i.e., (4.27) defines a harmonic function on M which is continuous on $M \cup S(\infty)$. On the other hand, if h is a harmonic function on M which is continuous in the whole $M \cup S(\infty)$, then by the probabilistic representation of solutions of the Dirichlet problem in the ball $B_x(r)$, $r > 0$ (cf. [KS]), $h(x) = E_x h(X(\tau_x(r)))$, and so by (4.22) and (4.26),

$$h(x) = \lim_{r \to \infty} E_x h(X(\tau_x(r))) = E_x h(X(\infty)) = \int_{S(\infty)} h(\zeta) P(x, d\zeta)$$

completing the proof of Corollary 4.6. ∎

The Hausdorff dimension of a set $U \subset S(\infty)$ corresponding to a pole x is given by

$$(4.29)\qquad \begin{aligned} Hd_x(U) &= \inf\left\{\delta \geq 0 : \lim_{\rho\to 0}\inf \sum_i \varphi_i^\delta = 0\right\} \\ &= \sup\left\{\delta \geq 0 : \liminf_{\rho\to 0} \sum_i \varphi_i^\delta = \infty\right\}, \end{aligned}$$

where the infimum inside the brackets is taken over all countable covers of U by the sets

$$V_x(\xi_i, \varphi_i) = \overline{\mathcal{C}_x(\xi_i, \varphi_i)} \cap S(\infty), \quad \xi_i \in S_x M, \quad 0 \leq \varphi_i \leq \rho$$

with the cones $\mathcal{C}(\xi_i, \varphi_i)$ defined before Lemma 2.6. The Hausdorff dimension of a probability measure μ on $S(\infty)$ is defined by

$$(4.30)\qquad HD_x(\mu) = \inf\{Hd_x(U) : U \subset S(\infty) \text{ and } \mu(U) = 1\}.$$

4.7. Proposition: *There exists a number $\kappa > 0$ and a sequence of numbers $C_\ell > 0$ such that*

$$(4.31)\qquad P_x\{\|\Xi_x(\infty,\omega) - \xi\| \leq \varphi \text{ and } \omega \in \Omega_x^{(\ell)}\} \leq C_\ell \varphi^\kappa$$

for any $x \in M$, $\xi \in S_x M$, and $\varphi \geq 0$ where $\Omega_x^{(\ell)}$, $\ell = 1, 2, \ldots$ is an increasing sequence of events such that $P_x(\bigcup_\ell \Omega_x^{(\ell)}) = 1$.

4.8. Corollary: *Harmonic measures $P(x, \cdot)$ have no atoms and, for any $x \in M$,*

$$(4.32)\qquad HD_x(P(x,\cdot)) \geq \kappa$$

where $\kappa > 0$ is the same as in Proposition 4.7.

This corollary follows from Proposition 4.7 by the same argument as in Corollary 3.1 from [KL], and so we will need to establish only Proposition 4.7 itself. For any $x \in M$, $\xi \in S_x M$, $t > 0$, and $\rho > 0$, define

$$U_t(x,\xi,\rho) = \bigcup\{B_{z_j}(2\rho) : 0 \leq j \leq [t(2\alpha^{-1} - \alpha/2)\rho^{-1}] + 1\}$$

where α is the same as in (3.2), the balls $B_{z_j}(2\rho)$ of radius 2ρ are centered at the points $z_j = \mathrm{Exp}_x((\frac{1}{2}\alpha t + j\rho)\xi)$, and $[\cdot]$ denotes the integral part.

4.9. LEMMA: *For any $\delta > 0$ there exists $\beta = \beta(\delta) > 0$ such that, for all $t > 0$,*

$$\begin{aligned} & P_x\{X(t) \in U_t(x, \Xi_x(\infty), t^\delta)\} \\ (4.33) \quad & \geq P_x\{\alpha t \leq R_x(t) \leq \alpha^{-1}t \text{ and } d_x^{R_x(t)}(X(t), \mathrm{Exp}_x(R_x(t)\Xi_x(\infty))) \leq t^\delta\} \\ & \geq 1 - \beta^{-1}t^{-\beta}. \end{aligned}$$

Proof: By (3.2) and (4.6) together with the Markov property of $X(t)$, it follows that the second probability in (4.33) is not less than (cf. (3.20) in [KL] and (4.21) in [BK])

$$\begin{aligned} & P_x\{\alpha t \leq R_x(t) \leq \alpha^{-1}t\} \\ & \quad \times \inf_{z \in B_x(\alpha^{-1}t) \setminus B_x(\alpha t)} P_z\{d_x^{d(x,z)}(z, \mathrm{Exp}_x(d(x,z)\Xi_x(\infty))) \leq (\alpha d(x,z))^\delta\} \\ & \geq (1 - \alpha^{-1}e^{-\alpha t})(1 - a^{-1}(\alpha^2 t)^{-a\delta}) \end{aligned}$$

provided $\delta \leq 1$ and $(\alpha^2 t)^\delta \geq a^{-1}$. If $\delta > 1$, then the left hand side of (4.33) becomes only bigger. For t satisfying $t < \alpha^{-2}a^{-1/\delta}$ we obtain (4.33) automatically by adjusting $\beta = \beta(\delta)$. ∎

Next, I am able to complete the proof of Proposition 4.7 similarly to [KL]. Fix $\delta = \frac{1}{2}$ and take $\beta = \beta(\frac{1}{2})$ as in Lemma 4.9. Put $t_\ell = \ell^{2/\beta}$, then (4.33) together with the Borel–Cantelli lemma yield that, for P_x-almost all $\omega \in \Omega$, there exists $\ell_x(\omega) < \infty$ measurably dependent on ω such that

$$(4.34) \qquad X(t_\ell, \omega) \in U_{t_\ell}(x, \Xi(\infty, \omega), t_\ell^{1/2}) \quad \text{for all } \ell \geq \ell_x(\omega).$$

Set $\Omega_x^{(\ell)} = \{\omega : \ell_x(\omega) \leq \ell\}$, $\ell = 1, 2, \ldots$. It follows from (2.9) that one can choose $L > 0$ so that, if $||\xi - \eta|| \leq e^{-Lt}$ for $\xi, \eta \in S_x M$, then

$$(4.35) \qquad d(\mathrm{Exp}_x(2\alpha t\xi), \mathrm{Exp}_x(2\alpha t\eta)) \leq 1.$$

Let $\varphi_\ell = e^{-Lt_\ell}$ and $\ell \geq j$; then by (4.34) and (4.35) for any $\xi \in S_x M$,

$$\begin{aligned} & P_x\{||\Xi(\infty, \omega) - \xi|| \leq \varphi_\ell \text{ and } \omega \in \Omega_x^{(j)}\} \\ (4.36) \quad & \leq P_x\{X(t_\ell) \in U_{t_\ell}(x, \xi, t_\ell^{1/2} + 1)\} \\ & \leq \sup_{x,y} p(t_\ell, x, y)\ \mathrm{vol}(U_{t_\ell}(x, \xi, t_\ell^{1/2} + 1)). \end{aligned}$$

Since the Ricci curvature of M is bounded from below, then by the volume comparison theorems (see [BC], Section 11.10 or [G], Section 3.5) the volume in the right hand side of (4.36) does not exceed $K\exp(Kt_\ell^{1/2})$ for some constant $K>0$ independent of x,ξ and ℓ. This together with (1.2) and (4.36) bounds the left hand side of (4.36) from above by

$$c^{-1}K\exp(Kt_\ell^{1/2}-ct_\ell)\le\tilde K\exp(-ct_\ell/2)=\tilde K\varphi_l^{c/2L}$$

for some $\tilde K>0$ independent of ℓ and, since $(\log\varphi_{\ell+1})(\log\varphi_\ell)^{-1}=t_{\ell+1}t_\ell^{-1}\to 1$ as $\ell\to\infty$, we derive (4.31). This completes the proof of Proposition 4.2, as well as Theorem B from Introduction. ∎

Similarly to Proposition 4.5 and Theorem 4.4 from [BK], one can establish also the following results:

4.10. PROPOSITION: *Suppose that with P_x-probability one the limit*

$$\lim_{t\to\infty}t^{-1}d(x,X(t))=\beta$$

exists and is not random, which holds, in particular, when M is the universal cover of a compact manifold. Then with probability one,

$$\lim_{t\to\infty}t^{-1}d(X(t),\mathrm{Exp}_x(\beta t\Xi_x(\infty)))=0.$$

4.11. PROPOSITION: *Suppose that M is the universal cover of a compact surface. Then $\Phi_x^{-1}P(x,\cdot)$ is singular with respect to the Lebesgue measure on S_xM unless the curvature is constant.*

4.12. Remark: The harmonic measures $P(x,\cdot)$ for different $x\in M$ are equivalent in view of the Harnack inequality (see, for instance, [LY]). ∎

4.13. Remark: In fact, $d_x^r(y,z)\le\rho$ for $x,y\in S_x(r)$ implies that $d(y,z)\le C\log\rho$ where C can be expressed via the hyperbolicity constant. Thus in the same way as in (4.33), one derives from (4.6) that with probability one $X(t)$ stays within the distance of order $\log t$ from the geodesic connecting x and $X(\infty)$ (cf. [An2]). ∎

4.14. Remark: The referee pointed out to me that one can improve (4.31) dropping $\omega\in\Omega_x^{(l)}$ and obtain a shorter analytic proof of the positivity of Hausdorff dimensions of harmonic measures in the following way. It follows from (2.7) of Lemma 2.2 that there is $c>0$ such that, if $y=\mathrm{Exp}_x(c\log(1/\varphi)\xi)$, $\xi\in S_xM$, then the distance between y and the geodesic ray emanating from x in any direction ζ, $\|\xi - \zeta\| \le \varphi$ does not exceed c^{-1}. Therefore, by the Harnack inequality at infinity from [An2] and by the exponential decay of the Green function $G(x,y)$, one derives for the harmonic function $h(x) = P_x\{\|\Xi_x(\infty,\omega) - \xi\| \le \varphi\}$ that $h(x) \le c_1 h(y) G(x,y) \le c_1 \varphi^\kappa$ for some $\kappa > 0$. Still, having in mind more general applications where the Green function is not available I gave my probabilistic proof here. ∎

5. The Martin boundary

By (1.2) the Green function

$$G(x,y) = \int_0^\infty p(t,x,y)dt > 0$$

exists and satisfies the following properties:

(i) $G(x,y) = G_y(x)$ is harmonic in $M \backslash \{y\}$ as a function of x;

(ii) $\Delta G_y = -\delta_y$ in the weak sense where δ_y denotes the Dirac measure at y;

(iii) $G(x,y) \le \beta^{-1} e^{-\beta d(x,y)}$ for $d(x,y) \ge 1$ where $\beta > 0$ is independent of x, y. For small $d(x,y)$, $G(x,y)$ behaves as $\log(d(x,y))^{-1}$ if $n = 2$ and it behaves as $(d(x,y))^{2-n}$ if $n > 2$.

In order to introduce the Martin boundary of M one needs first the uniform Harnack inequality (in fact, the uniformity is not necessary here) saying that for any $r_2 > r_1 > 0$ there exists a constant C^{r_1,r_2} such that, for any $x \in M$ and a harmonic in $B_x(r_2)$ function $h > 0$,

$$\sup\{h(y) | y \in B_x(r_1)\} \le C^{r_1,r_2} \inf\{h(y) | y \in B_x(r_1)\}. \tag{5.1}$$

This inequality under the condition of boundedness of the Ricci curvature from below follows from the even more general parabolic Harnack inequality proved in [LY]. Furthermore, by the gradient estimates in [LY] it follows that there exists $C > 0$ such that, for any harmonic $h > 0$,

$$|\nabla h| \le Ch. \tag{5.2}$$

Now the Ascoli–Arzela theorem together with (5.1) and (5.2) give rise to the Harnack principle: A sequence of positive harmonic functions h_j in $B_x(r)$ has a uniformly convergent subsequence provided there exist $y \in B_x(r)$ and $\tilde{C} > 0$

such that $h_j(y) \le \tilde{C}$ for all $j = 1, 2, \ldots$. The limiting function must be harmonic in view, for instance, of the probabilistic characterisation of harmonic functions in $B_x(r)$ as functions h satisfying $h(y) = E_y h(\tau_x(r))$ which represents also the solution of the Dirichlet problem in $B_x(r)$ with the boundary data equal to h on $S_x(r)$. Replacing here h by h_j and assuming that $h_j \to h$ as $j \to \infty$, we see that h must satisfy the same relation, and so h is also harmonic.

Fix a point $p \in M$ and set

$$K_y(x) = K(x,y) = \begin{cases} \frac{G(x,y)}{G(p,y)} & \text{if } y \ne p, \\ 0 & \text{if } y = p, x \ne p, \\ 1 & \text{if } x = y = p, \end{cases}$$

which is called the Martin function. Let $\{y_i\}$ be a sequence in M having no limit points in M and choose an increasing sequence of balls $B^{(i)}$ such that $y_j \notin \overline{B^{(i)}}$ for all $j \ge i$. Then the functions K_{y_j} are harmonic in $B^{(i)}$ for $j \ge i$. Since $K_{y_j}(p) = 1$, then by the Harnack inequality the functions $K_{y_j}, j \ge i$ are uniformly bounded in $B^{(i)}$. The sequence $\sigma = \{y_i\}$ is called fundamental if $\{K_{y_i}\}$ converges to a harmonic function K_σ on M. By the Harnack principle any sequence $\{y_i\}$ having no limit points in M has a fundamental subsequence. Fundamental sequences corresponding to the same limit harmonic function form an equivalence class. The Martin boundary ∂M of M is defined to be the set of equivalence classes of fundamental sequences. If $[\sigma] \in \partial M$, then $K_\sigma = \lim_{i\to\infty} K_{y_i}$, where $\{y_i\}$ is a fundamental sequence associated to $[\sigma]$ and so points $[\sigma] \in \Delta$ correspond uniquely to certain positive harmonic functions K_σ on M. Put $\mathcal{M} = M \cup \partial M$. There are many equivalent ways of defining a metric on $\mathcal{M}$. For instance, one can define the distance between $\sigma_1, \sigma_2 \in \mathcal{M}$ by

$$(5.4) \qquad \rho(\sigma_1,\sigma_2) = \sum_{j=1}^{\infty} 2^{-j} \sup_{x\in B_p(j)} \left(\frac{|K_{\sigma_1}(x) - K_{\sigma_2}(x)|}{1 + |K_{\sigma_1}(x) - K_{\sigma_2}(x)|} \right).$$

It follows from [An2] together with Theorem A that there is a homeomorphism $\varphi : \partial M \to S(\infty)$ establishing the first part of Theorem C. One obtains also the unique representation of any positive harmonic function h as an integral $h = \int_{S(\infty)} K_{\varphi_\sigma} d\mu_h(\sigma)$ with μ_h being a probability measure on ∂M, and a representation of any bounded harmonic function h as an integral $h(x) = \int_{S(\infty)} f_h(\zeta) P(x, d\zeta)$ where f_h is a Borel function on $S(\infty)$ unique up to a set of harmonic measure zero. The proof in [An2] for the general hyperbolic case employs estimates for the Green function and for positive harmonic functions in certain domains defined via Gromov's "product" of points. These domains can be quite complicated and they do not help if one wants estimates of harmonic functions, such as the Harnack inequality at infinity, in cones, which is natural in our circumstances and is necessary in order to obtain a natural Hölder structure on ∂M. In order to do this I will assume throughout the rest of this section that M is a generalized CH-manifold with the Ricci curvature bounded from below satisfying the Uniform Visibility Axiom. In this case, in view of (2.2), Proposition 2.5(i), and Lemma 2.6 one can construct Φ-chains needed in [An1] and [An2] consisting of cones, and so the machinery of these papers will work here, as well. Nevertheless, I will review here the probabilistic approach from [Ki]. Set $\mathcal{T}_p(\xi,\theta,r)=\mathcal{C}_p(\xi,\theta)\backslash B_p(r)$.

5.1. LEMMA: *For any positive $\theta<\pi$ there exists $\beta=\beta(\theta)>0$ such that for any $p\in M,\xi\in S_pM$, and every positive harmonic function h defined in the cone $\mathcal{C}_p(\xi,\theta)$ and vanishing continuously on $\overline{\mathcal{C}_p(\xi,\theta)}\bigcap S(\infty)$,*

$$h(x)\le\beta^{-1}e^{-\beta d(x,p)}h(\mathrm{Exp}_p\left(\beta^{-1}\xi\right)) \tag{5.5}$$

provided $x\in\mathcal{T}_p\left(\xi,\theta/2,\beta^{-1}\right)$.

Proof: I proceed similarly to Lemma 4.1 in [Ki] with a modification needed in our more general situation. Set $d=d(x,p)$ and $r(j)=L^{j-1}d, j=0,1,2,\dots$ where $L=L(\theta)\ge 2$ is a big constant which will be specified below. Define $\partial_0^+=S_p(d)\bigcap\mathcal{C}_p(\xi,\theta/2),\partial_1^-=S_p(L^{-1}d)\bigcap\mathcal{C}_p(\xi,2\theta/3),l_1=\pi_p^d\partial_1^-,\partial_1^+=\pi_p^{Ld}\partial_1^-$, and for $j=1,2,\dots$ by induction

$$\begin{aligned}\partial_{j+1}^-&=\left\{y\in S_p\left(r(j)\right):d_p^{r(j)}(y,l_j)\le 2^{j^2}d\right\},\\ l_{j+1}&=\pi_p^{r(j+1)}\partial_{j+1}^-,\partial_{j+1}^+=\pi_p^{r(j+2)}\partial_{j+1}^-,\text{ and}\\ D_j&=\bigcup_{r(j-1)\le u\le r(j+1)}\pi_p^u\partial_j^-.\end{aligned}$$

By (2.2) and Proposition 2.5 (i), if $\beta>0$ is chosen small enough and $d\ge\beta^{-1}$ then

$$\bigcup_j D_j\subset\mathcal{C}_p(\xi,5\theta/6). \tag{5.6}$$

This construction is similar to what I had in Proposition 4.2 but the parameters are different. Let τ_j be the first exit time of $X(t)$ from D_j. Then employing the probabilistic representation of solutions of the Dirichlet problem (see, for instance, [KS]) in D_j one can write, for $j \geq 1$ and $y \in \partial^+_{j-1}$,

$$(5.7)\qquad \begin{aligned} h(y) &= \int_{\partial D_j} P_y\{X(\tau_j) \in dz\} h(z) \\ &\leq \sup_{z\in\partial_j^+} h(z) + P_y\{\tau_p(r(j-1)) < \tau_p(r(j+1))\} \sup_{z\in\partial_j^-} h(z) \\ &\quad + Q_y\left(p; r(j-1), r(j-1), r(j+1), 2^{(j-1)^2} d\right) \sup_{z\in\partial_j^s} h(z) \end{aligned}$$

where $\partial_j^s = \partial D_j \backslash (\partial_j^+ \bigcup \partial_j^-)$ and Q_y is the same as in (4.1). If $\beta = \beta(\theta)$ is chosen small enough, then any pair of points $z, \tilde{z} \in \mathcal{T}_p(\xi, 5\theta/6, \beta^{-1})$ can be connected by a chain $z_i, i = 0, 1, \ldots, k$ with $z_0 = z, z_{k(d)} = \tilde{z}, d(z_i, z_{i+1}) = \frac{1}{2}, B_{z_i}(1) \subset \mathcal{C}_p(\xi, \theta)$ for all i, and $k \leq C(\theta) d(z, \tilde{z})$ where $C(\theta) > 0$ is independent of $z, \tilde{z}$. This together with the Harnack inequality (5.1) yields that there exists $\delta = \delta(\theta) > 0$ such that

$$(5.8)\qquad \delta e^{\delta d(z,p)} \leq h(z)\left(h\left(\mathrm{Exp}_p\left(\beta^{-1}\xi\right)\right)\right)^{-1} \leq \delta^{-1} e^{\delta^{-1} d(z,p)}.$$

Choosing L larger than $2 + 4\alpha^{-1}\delta^{-1}$, one derives from (3.14), (3.15), (4.1), and (5.6)–(5.8) that there exists $\kappa = \kappa(\theta) > 0$ such that

$$(5.9)\qquad \sup_{y\in\partial^+_{j-1}} h(y) \leq \sup_{z\in\partial_j^+} h(z) + \kappa^{-1} e^{-\kappa r(j)}.$$

Since h vanishes continuously at infinity, i.e.

$$\lim_{j\to\infty} \sup_{z\in\partial_j^+} h(z) = 0,$$

then applying (5.9) for $j = 1, 2, \ldots$ we derive (5.5). ∎

Next, one defines a kernel function k_ζ at $\zeta \in S(\infty)$ as a positive harmonic function on M having a continuous extension into $M \bigcup (S(\infty)\backslash\zeta)$ with zero values on $S(\infty)\backslash\zeta$. Lemma 5.1 together with the property (iii) of the Green function enables us to produce kernel functions at each $\zeta \in S(\infty)$.

5.2. COROLLARY: *Let $\{y_i\}$ be a sequence of points with $y_i \longrightarrow \zeta \in S(\infty)$ in the cone topology as $i \to \infty$. Then any fundamental subsequence $\sigma = \{y_{i_j}\}$ gives rise to a kernel function $k = K_\sigma$ at ζ. All fundamental sequences from the* *equivalence class $[\sigma] \in \Delta$ determine kernel functions at the same point ζ. This defines a map $\varphi : \partial M \longrightarrow S(\infty)$ which is a continuous surjection.*

The proof of this result is the same as in Corollary 4.1 in [Ki]. In order to complete the identification of the Martin boundary one needs the following result:

5.3. LEMMA: *For any positive $\theta < \pi$ there exists $C = C(\theta) > 0$ such that for any $p \in M, \xi \in S_pM$, and every positive harmonic function h defined in $\mathcal{C}_p(\xi,\theta)$ and vanishing continuously on $\overline{\mathcal{C}_p(\xi,\theta)} \bigcap S(\infty)$,*

$$C^{-1}(\theta)\rho_\theta(x) \le h(x)\left(h\left(\operatorname{Exp}_p(\xi)\right)\right)^{-1} \le C(\theta)\rho_\theta(x) \tag{5.10}$$

provided $x \in \mathcal{T}_p(\xi,\theta/2,C)$, where ρ_θ is a positive function defined in $\mathcal{T}_p(\xi,\theta/2,C)$ and independent of h.

Proof: One proceeds in the same way as in Lemma 4.2 from [Ki] with modifications in the construction of domains below similar to the changes in Lemma 5.1 with respect to Lemma 4.1 from [Ki]. The construction depends on numbers $L = L(\theta), C = C(\theta), d = d(x,p)$, and $k = k(\theta,d)$ satisfying $L^{-k}d = C(\theta)$ with L and C large enough. Set $r(j) = L^{-j+2}d$ for $j = 0,\dots,k+2, \partial^-_{k+1} = S_p(r(k+2)) \bigcap \mathcal{C}_p(\xi,5\theta/6)$, $l_{k+1} = \pi_p^{r(k+1)}\partial^-_{k+1}$, and $\partial^+_{k+1} = \pi_p^{r(k)}\partial^-_{k+1}$. If $\partial^-_j, l_j, \partial^+_j$ are already defined then I define ∂^-_{j-1} by the equality

$$l_j = \left\{y \in S_p(r(j+1)) : d_p^{r(j+1)}\left(y,\partial^-_{j-1}\right) \le 2^{(k-j)^2}\right\},$$

$$l_{j-1} = \pi_p^{r(j)}\partial^-_{j-1}, \partial^+_{j-1} = \pi_p^{r(j-1)}\partial^-_{j-1}, \quad \text{and} \quad D_j = \bigcup_{r(j+1)\le u\le r(j-1)} \pi_p^u\partial^-_j.$$

Now the domains $D_1,\dots,D_{k+1}$ widen in the direction from x to p, and not away from p as in Lemma 5.1. Choosing L and C large enough I can have, by (2.2) and Proposition 2.5(i), that

$$\partial^+_1 \supset S_p(L^2d) \bigcap \mathcal{C}_p(\xi,\theta/2). \tag{5.11}$$

Now writing for $y \in \partial^-_{j-1}, j = 2,\dots,k+1$,

$$h(y) = \int_{\partial D_j} P_y\{X(\tau_j) \in dz\}h(z),$$

estimating h on ∂_j^+ by (5.5) and on $\partial_j^s = \partial D_j \setminus (\partial_j^+ \bigcup \partial_j^-)$ by (5.8) and estimating the probabilities of the events $X(\tau_j) \in \partial_j^-, X(\tau_j) \in \partial_j^+, X(\tau_j) \in \partial_j^s$ by (3.14) and (4.1) one obtains (5.10) in the same way as in Lemma 4.2 from [Ki] with

$$q_\theta(x) = \int_{\partial_1^-} P_x\{X(\tau_1) \in dz_1\} \cdots \int_{\partial_{k+1}^-} P_{z_k}\{X(\tau_{k+1}) \in dz_{k+1}\}. \qquad (5.12)$$

∎

Lemma 5.3 implies the following Harnack inequality at infinity.

5.4. COROLLARY: *If two functions, positive harmonic in $\mathcal{C}_p(\xi,\theta)$, $\theta > 0$, h_1 and h_2, vanish continuously on $\overline{\mathcal{C}_p(\xi,\theta)} \bigcap S(\infty)$, then*

$$\sup_{x\in\mathcal{T}} \frac{h_1(x)}{h_2(x)} \le C^4 \inf_{x\in\mathcal{T}} \frac{h_1(x)}{h_2(x)} \qquad (5.13)$$

where $\mathcal{T} = \mathcal{T}_p(\xi, \frac{\theta}{2}, C)$ and $C = C(\theta)$ is the same as in (5.10).

5.5. COROLLARY: *The map $\varphi : \partial M \longrightarrow S(\infty)$ introduced in Corollary 5.2 is a homeomorphism.*

Proof: It remains to show that φ is one-to-one. Suppose that h_1 and h_2 are two kernel functions at $\zeta \in S(\infty)$ such that $h_1(p) = h_2(p) = 1$. Let $\xi \in S_pM$, $p_t = \mathrm{Exp}_p(t\xi)$, and $\lim_{t\to\infty} p_t = \zeta$. Set $\mathcal{C}_t = \mathcal{C}_{p_t}(\xi_t, \frac{\pi}{4})$ where $\xi_t \in S_{p_t}M$ satisfies $\mathrm{Exp}_{p_t}(s\xi_t) = p_{t+s}$. By (2.16),

$$\mathcal{C}_t \subset \mathcal{C}_{t+r(\pi/4)} \quad \text{and} \quad \bigcap_{j\ge 0} \mathcal{C}_{t+jr(\pi/4)} = \emptyset \qquad (5.14)$$

where $r(\varepsilon)$ is the number from the Uniform Visibility Axiom. Proceeding as in the proof of Theorem 4.1 from [Ki] (see also [AS] and [An1]) we derive from (5.13) and (5.14) that the ratio h_1/h_2 is sandwiched between two positive constants independent of h_1 and h_2, which in turn implies that $h_1 = h_2$. ∎

It remains to construct a natural Hölder structure on ∂M and show that both φ and φ^{-1} are Hölder continuous, which means essentially that there exists $\delta > 0$ such that, if k_{ζ_1}, k_{ζ_2} are two kernel functions at $\zeta_1, \zeta_2 \in S(\infty)$ and $\xi_1, \xi_2 \in S_pM$ satisfy $\lim_{t\to\infty} \mathrm{Exp}_p(t\xi_i) = \zeta_i, i = 1, 2$, then

$$|k_{\zeta_1}(x) - k_{\zeta_2}(x)| \le \delta^{-1} \|\xi_1 - \xi_2\|^\delta. \qquad (5.15)$$

The proof proceeds in exactly the same way as in Theorems 6.2 and 6.3 from [AS] taking into account that by (2.16), for any $p \in M$ and $\xi \in S_pM$,

$$\mathcal{C}_{\gamma_\xi(t)}\left(\dot\gamma_\xi(t), \frac{\pi}{4}\right) \subset \mathcal{C}_{\gamma_\xi(s)}\left(\dot\gamma_\xi(s), \frac{\pi}{8}\right) \quad \text{if } t \geq s + r\left(\frac{\pi}{8}\right). \tag{5.16}$$

In addition to this, the proof in [AS] needs only the following geometrical statement saying that if

$$\mathcal{C}_j = \mathcal{C}_{\gamma_\xi(jr(\pi/8))}\left(\dot\gamma_\xi\left(jr\left(\pi/8\right)\right), \pi/4\right), \dot\gamma_\xi(0) = \xi,$$

then there exists $K > 0$ such that, for all $\xi \in S_pM$ and j large enough,

$$\angle_p(\mathcal{C}_j) \geq e^{-Kj}. \tag{5.17}$$

But by Proposition 2.5(i) if $r_j = (j+1)r(\pi/8) + 1$, then

$$\mathcal{C}_j \bigcap S_p(r_j) \supset \left\{z \in S_p(r_j) : d_p^{r_j}\left(\gamma_\xi(r_j), z\right) < 1\right\},$$

and (5.17) follows by (2.9). The remaining part of the proof is the same as in [AS]. The proof of Theorem C is now complete. ∎

6. Concluding remarks

I will discuss here two types of generalizations: (i) when the "no focal points" condition is replaced by the "no conjugate points" assumption; (ii) when the hyperbolicity of the manifold is replaced by some other condition. So assume that M is a complete simply connected hyperbolic manifold without conjugate points (see, for instance, [Eb1]). Without any further conditions one does not know whether there are different geodesic rays with a common origin which are asymptotic to each other. Still, if we assume, in addition, that M satisfies the Visibility Axiom, then for any two geodesics γ_ξ, γ_η with $\gamma_\xi(0) = \gamma_\eta(0) = p, \xi, \eta \in S_pM, \xi \neq \eta$ one has $\lim_{t\to\infty} d(\gamma_\xi(t), \gamma_\eta(t)) = \infty$. Indeed, if $t_n \longrightarrow \infty$ and $d(\gamma_\xi(t_n), \gamma_\eta(t_n)) \leq C$ then the geodesic γ_n connecting $\gamma_\xi(t_n)$ and $\gamma_\eta(t_n)$ satisfies $\angle_p(\gamma_n) = \angle_p(\xi, \eta) > 0$ and $d(p, \gamma_n) \geq t_n - \frac{1}{2}C \longrightarrow \infty$ as $t_n \to \infty$, contradicting the Visibility Axiom. So in this case the geometric boundary $S(\infty)$ is homeomorphic to the sphere S_pM.

Next, (2.5) fails to be true in general and, in order to obtain (2.6)–(2.9) and (2.17), one has to assume that the sectional curvature rather than the Ricci curvature is bounded from below. In order to have some lower bound on $Q_x(r,\xi)$, which does not have to be positive now, one needs also an upper bound on the sectional curvature of M. Under these assumptions I can derive again the conclusion of Lemma 2.14. But this does not imply Proposition 3.1, which relies heavily on the fact that though one may not have the lower bound (2.42) at any point, still in the "no focal points" case $Q_x(r,\xi)$ is nonnegative (in fact, positive) everywhere. I can overcome this difficulty only in the case when M is a universal cover of a compact manifold N or under an additional assumption (6.1) below. In the first case it is not difficult to see that there exist $\delta > 0$ such that the integral of $Q_x(r,\xi)$ against the volume over any fundamental domain D is at least δ provided $d(x,D) \geq \delta^{-1}$. This follows similarly to Lemma 2.14 taking into account that for large r, $Q_x(r,\xi)$ is close to the trace of the operator of the second fundamental form of a corresponding horosphere which is a function on N. Thus the integrals of Q_x over fundamental domains which are far away from x are almost the same and the assertion follows similarly to (2.44)–(2.50). Now taking into account that the Lebesgue measure on N is invariant with respect to the Brownian motion on N and that $X(t)$ is its lifting to M, we can conclude by the ergodic theorem that $E_y\int_0^t Q_x(X(s))ds$ is positive and it is bounded away from zero provided t is large enough. This is good enough to obtain the results of Sections 3 and 4 (see below). Note that the weak coercivity of the Laplace–Beltrami operator follows in this case also from [Br], and so Theorem C except for the Hölder structure follows from [An2]. M. Gromov suggested that, using local isoperimetric inequalities together with the global divergence provided by the hyperbolicity, one may be able to prove Theorem A under the "no conjugate points" condition provided M is hyperbolic which was fulfilled recently in [Cao2]. By [An2], this implies the first part of Theorem C under these assumptions. In this direction I can do by the probabilistic method the following. Suppose that there exists $L > 0$ such that, for any $x, y \in M$,

$$(6.1)\qquad E_y\int_0^L Q_x(X(s))ds = \int_0^L\int_M p(s,y,z)Q_x(z)dm(z)ds \geq L^{-1}.$$

It is easy to see from Theorem 4 of [F] that (6.1) implies (3.2). Proceeding in the same way as in Sections 3–5 we derive from here the following result:

6.1. THEOREM: *Let M be a complete, simply connected, n-dimensional, $n \geq 2$, C^3 Riemannian manifold without conjugate points and with sectional curvature* *bounded above and below with (6.1) being satisfied. Then (1.2) holds true, and so the top of the L^2-spectrum of Δ is negative. Suppose, in addition, that M is hyperbolic and satisfies the Visibility Axiom; then the conclusions of Theorems B and C hold true as well (assuming the Uniform Visibility Axiom for the Hölder structure on ∂M).*

The condition (6.1) depends, of course, only on the geometry of M but I do not know how to express it in purely geometric terms. On the other hand, it is clear (and important) that (6.1) is preserved under a small perturbation of the metric. Note that (6.1) follows from either of the following two conditions:

$$\int_{L_1}^{L_2} \int_M p(s,y,z) Q_x(z) dm(z) ds \geq \delta \tag{6.2}$$

for some $L_2 > L_1 > 0$ and $\delta > 0$ independent of $x, y \in M$, or

$$\int_M p(L,y,z,) Q_x(z) dm(z) ds \geq \delta \tag{6.3}$$

for some $L, \delta > 0$ independent of $x, y \in M$. Indeed, by the semigroup property of the heat kernel $p(t,x,y)$, (6.2) yields

$$\int_{L_1}^{L_1+n(L_2-L_1)} \int_M p(s,y,z) Q_x(z) dm(z) ds \geq n\delta, \tag{6.4}$$

and so (6.1) will be satisfied if $n\delta > -L_1 \inf_{x,z} Q_x(z)$. If (6.3) holds true, then multiplying both parts of (6.3) by $p(s,v,y)$ and integrating in y against $dm(y)$ we obtain, by the semigroup property of the heat kernel, that (6.3) remains true for any $t \geq L$ in place of L which implies (6.2) with $L_1 = L$ and $L_2 = L+1$.

By (2.4), $Q_x(z) = \Delta_z \rho_x(z)$, $\rho_x(z) = d(x,z)$ where Δ_z acts in the z-variable. Thus integrating in (6.1) by parts and taking into account that $p(s,y,z) = p(s,z,y)$ and $\Delta_z p(s,z,y) = \partial p(s,z,y)/\partial s$, we obtain

$$\begin{aligned}\int_0^L \int_M p(s,y,z) Q_x(z) dm(z) ds &= \int_M p(L,y,z) \rho_x(z) dm(z) - \rho_x(y) \\ &= E_y d(x, X(L)) - d(x,y).\end{aligned} \tag{6.5}$$

Thus, the condition (6.1) is equivalent to

$$E_y d(x, X(L)) \geq d(x,y) + L^{-1} \tag{6.6}$$

with some $L > 0$ independent of $x, y \in M$. Of course, (6.6) follows from (3.2), i.e. it holds true under the "no focal points" condition. Actually, (6.3) (and so (6.1), (6.2), and (6.6), as well) follows from (2.5), (2.42), and (3.4). It seems plausible that the condition (6.1) (and so (6.6)) is always satisfied for some $L > 0$ in the no conjugate points hyperbolic set up, provided M has a lower bound on the Ricci curvature and a bounded geometry.

6.2. Remark: The condition (6.6) makes sense even when conjugate points are allowed and, by Theorem 4 from [F], it always implies (3.2) with $R_x(s) = d(x, X(s))$. These yield Theorem A assuming (6.6) without the "no conjugate points" condition. If, in addition, M is hyperbolic and satisfies the Visibility Axiom, then the conclusions of Theorems B and C hold true as well. Condition (6.6) is so general that it enables one to prove similar results for certain class of Markov processes in general hyperbolic geodesic metric spaces which will be discussed in another paper. ∎

Another direction of research is when one does not have an exponential divergence of geodesics provided by the hyperbolicity of M. Suppose, for instance, that M is a CH-manifold of rank 1 (see [Ba]). If M has a compact quotient N, then by [Bal] the Dirichlet problem at infinity has a unique solution. Moreover, Ballmann and Ledrappier proved in this case that the Poisson boundary of M coincides with $S(\infty)$, i.e. any bounded harmonic function on M can be represented as an integral of type (1.1) with some Borel function f on $S(\infty)$. Nevertheless, it is not known whether the Hausdorff dimensions of harmonic measures are positive and what is the Martin boundary of M. The main difficulty with our probabilistic method in this situation is connected with flat planes in M which makes it hard to estimate the angular shift of the Brownian motion. Still, when one has a good control of these planes, say, one has only one flat plane as in [Ba2], then our approach works well enough. For instance, writing explicitly the Laplace–Beltrami operator for the example from [Ba2] we see that the Brownian motion has a positive drift away from the flat plane, and so gets quickly to the region of negative curvature where our methods do work. Nevertheless, it does not seem plausible that one can define in this direction a sufficiently general class of manifolds where the behavior of the Brownian motion and the description of spaces of harmonic functions is similar to the negative curvature case. Clearly, it is not sufficient to assume just that M is rank 1 without an additional assumption that M has a compact quotient N, since if M has a point where all sectional curvatures are negative then M has already rank 1 but, in general, one point and even a compact set does not influence the structure of harmonic functions on M.

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